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Relative torsion theories

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Abstract

We formulate necessary and sufficient conditions for a pair of subcategories to form a relative torsion theory.

Key words: reflective and coreflective subcategories, relative torsion theories, the right and left product of two subcategories, locally convex spaces.

1 Introduction

In the category $\mathcal{C}_2\mathcal{V}$ of the vectorial topological locally convex Hausdorff spaces we examined the subcategories: Γ_0 – the subcategory of the complete spaces, $\mathcal{S}$ – the subcategory of the spaces with weak topology, $\widetilde{\mathcal{M}}$ – the subcategory of the spaces with the Mackey topology (see [4]); the classes of morphisms: $\mathcal{M}_u$ – the class of universal monomorphisms (see [2]), $\mathcal{E}pi$ – the class of epimorphisms; $\mathcal{M}ono$ – the class of monomorphisms, $\mathcal{I}so$ – the class of isomorphisms, if $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ (respectively: $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$) is a reflector functor (respectively: coreflector), then: $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so\}$, $\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid k(m) \in \mathcal{I}so\}$. The factorization structures $(\mathcal{E}_u, \mathcal{M}_p)$, $(\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K}))$, $(\mathcal{P}''(\mathcal{R}), \mathcal{P}''(\mathcal{R}))$ are described in [2], the right and left product of two subcategories are described in [3].

2 The right and left product of two subcategories and the relative torsion theories

Definition 1 [1]. *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ be a reflective subcategory of category $\mathcal{C}$. The pair $(\mathcal{K}, \mathcal{R})$ is called a relative* *torsion theory (RTT), i.e. relative to the subcategory* $\mathcal{K} \cap \mathcal{R}$*, if the functors* $k : \mathcal{C} \to \mathcal{K}$ *and* $r : \mathcal{C} \to \mathcal{R}$ *verify the following two relations:*

1. *The functors* k *and* r *commute:* $k \cdot r = r \cdot k$*;*

2. *For any object* X *of category* $\mathcal{C}$ *the square* $r^X \cdot k^X = k^{rX} \cdot r^{kX}$ *is pull-back and puschout, where* $k^X : kX \to X, k^{rX} : krX \to rX$ *are the* $\mathcal{K}$*-coreplique, and* $r^X : X \to rX$ *and* $r^{kX} : kX \to rkX = krX$ *are* $\mathcal{R}$*-replique of the respective objects.*

Remark 1. *In the abelien categories a torsion theory* $(\mathcal{T}, \mathcal{F})$ *is a RTT relative to the intersection* $\mathcal{T} \cap \mathcal{F} = 0$*.*

Theorem 1 ([1]). *Let* $\mathcal{K}$ *be a coreflective subcategory, and* $\mathcal{R}$ *– a reflective subcategory of category* $\mathcal{C}_2\mathcal{V}$ *and* $\Gamma_0 \subset \mathcal{R}$*. The pair* $(\mathcal{K}, \mathcal{R})$ *forms a RTT iff the coreflector functor* $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ *and the reflector* $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ *commute:* $k \cdot r = r \cdot k$*.*

Remark 2. *Examples of RTT and coreflective and reflective functors that commute, can be found in* [1].

Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ be a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$. We examine the following conditions:

(S) The subcategory $\mathcal{K}$ is closed with respect to $(\varepsilon\mathcal{R})$-factorobjects.

(D) The subcategory $\mathcal{R}$ is closed with respect to $(\mu\mathcal{K})$-subobjects.

Lemma 1. *The subcategory* $\mathcal{R}$ *has the property (D), if for any object* (E, u) *and every locally convex topology* v *with properties* $u \leq v \leq k(u)$*, where* $(E, k(u))$ *is* $\mathcal{K}$*-coreplique of object* (E, u)*, the object* (E, v) *also belongs to subcategory* $\mathcal{R}$*.*

Lemma 2. *For the subcategories* $\mathcal{K}$ *and* $\mathcal{R}$ *of category* $\mathcal{C}_2\mathcal{V}$ *the following affirmations are equivalent:*

1. $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$.

2. *The subcategory* $\mathcal{K}$ *satisfies the condition (S).*

If the subcategory $\widetilde{\mathcal{M}} \subset \mathcal{K}$*, then the previous conditions are equivalent to the condition:*

3. *The subcategory* $\mathcal{K}$ *is closed with respect to* $\mathcal{P}''(\mathcal{R})$*-factorobjects.*

Dual statement.

Lemma 3. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$ the following conditions are equivalent:*

1. $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

2. *The subcategory $\mathcal{R}$ satisfies the condition (D).*

If $\mathcal{S} \subset \mathcal{R}$, then the previous conditions are equivalent to the condition:

3. *The subcategory $\mathcal{R}$ is closed with respect to $\mathcal{M}'(\mathcal{K})$-subobjects.*

Theorem 2. *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ – a reflective subcategory. The following statements are equivalent:*

1. *The pair $(\mathcal{K}, \mathcal{R})$ forms a RTT.*

2. a) *The functors k and r commute: $k \cdot r = r \cdot k$;*

b) $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$;

c) $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

3. a) *The functors k and r commute: $k \cdot r = r \cdot k$;*

b) *The subcategory $\mathcal{K}$ possesses the property (S);*

c) *The subcategory $\mathcal{R}$ possesses the property (D).*

If $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ then the previous conditions are equivalent to the following:

4. a) *The functors k and r commute: $k \cdot r = r \cdot k$;*

b) *The subcategory $\mathcal{K}$ is closed with respect to $\mathcal{P}''(\mathcal{R})$-factorobjects;*

c) *The subcategory $\mathcal{R}$ is closed with respect to $\mathcal{M}'(\mathcal{K})$-subobjects.*

Theorem 3. *Let it be $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\Gamma_0 \subset \mathcal{R}$. Then:*

1. *The subcategory $\mathcal{K}$ is closed with respect to $(\mathcal{E}pi \cap \mathcal{M}_p)$-factorobjects. In other words, the subcategory $\mathcal{K}$ is closed with respect to extensions.*

2. *The subcategory $\mathcal{R}$ is closed with respect to $(\mu\widetilde{\mathcal{M}})$-subobjects. In other words, if the locally convex spaces (E, t) belong to the subcategory $\mathcal{R}$, then the space E belongs to the subcategory $\mathcal{R}$ with every locally convex topology u stronger than t, but compatible with the same duality: $t \leq u \leq m(t)$, where $(E, m(t))$ is the $\widetilde{\mathcal{M}}$-coreplique of the object (E, t).*

Remark 3. 1. *For some subcategories $\mathcal{K}$ with the property $\widetilde{\mathcal{M}} \subset \mathcal{K}$, in particular, for the subcategory $\widetilde{\mathcal{M}}$, it is well known that they are closed with respect to extensions* ([4], Affirmation IV.3.5.).

2. *Every locally convex complete space* (E, t) *remains complete in any topology* u *stronger than* t *but compatible with the same duality:* $t \leq u \leq m(t)$ ([4], VI Corollary of Proposition 3).

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