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About one special inversion matrix of 3-ary and 4-ary IP-loops

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Abstract

It is known that an n-IP-quasigroup can have more than one inversion matrix. We prove that one of these matrices for a 3-ary IP-loop and for a 4-ary IP-loop is the matrix of permutations every of which fixes identity of a loop and has order two. It is a good matrix allowing to investigate n-IP-loops in more detail.

Keywords: 3-IP-loop, 4-IP-loop, inversion system, inversion matrix

1 Introduction

The definitions of an n-IP-quasigroup (an n-IP-loop), $n \geq 2$, and of its inversion matrix one can find in [1]. It is known that an n-IP-quasigroup can have more than one inversion matrix and one of these matrices for an n-IP-loop with an identity e can be a matrix of the special form $[I_{ij}]_e$ which facilitates the study of n-IP-loops. V.D.Belousov has assumed that every n-IP-loop has the matrix $[I_{ij}]_e$ as an inversion matrix.

In [2], the example of a 3-IP-loop for which one of inversion matrices is the matrix $[I_{ij}]_e$ was given. Later in [3], it was proved that the matrix $[I_{ij}]_e$ exists for any n-IP-group with an identity e, for any symmetric n-IP-loop and for any n-IP-loop with one inversion parameter. The existence of the matrix $[I_{ij}]_e$ for any nonsymmetric n-IP-loops at present is not proved.

In this article we establish that the matrix $[I_{ij}]_e$ exists for any nonsymmetric 3-ary IP-loop and for any nonsymmetric 4-ary IP-loop with an identity e.

2 Preliminaries

A ternary operation $Q(\,)$, defined on a set Q, is called *a 3-ary quasigroup with the invertible property (shortly, a 3-IP-quasigroup)* if on Q there exist permutations υ_{ij}, $i = 1,2,3$ (or $i \in \overline{1,3}$), $j \in \overline{1,4}$, where $\upsilon_{ii} = \upsilon_{i4} = \varepsilon$ (ε is the identity permutation) such that the following equalities hold: $((x_1^3), \upsilon_{12}x_2, \upsilon_{13}x_3) = x_1$, $(\upsilon_{21}x_1, (x_1^3), \upsilon_{23}x_3) = x_2$, $(\upsilon_{31}x_1, \upsilon_{32}x_2, (x_1^3)) = x_3$ for any $x_1^3 \in Q^3$.

The matrix

$$[\upsilon_{ij}] = \begin{bmatrix} \varepsilon & \upsilon_{12} & \upsilon_{13} & \varepsilon \\ \upsilon_{21} & \varepsilon & \upsilon_{23} & \varepsilon \\ \upsilon_{31} & \upsilon_{32} & \varepsilon & \varepsilon \end{bmatrix}$$

is called *an inversion matrix* for a 3-IP-quasigroup, the permutations $\upsilon_{i,j}$ are called *inversion permutations.* Any row of an inversion matrix is called *an inversion system* for a 3-IP-quasigroup.

All these definitions are analogous for any n-IP-quasigroups.

An n-IP-quasigroup is called *symmetric or an n-TS-quasigroup* if $\upsilon_{ij} = \varepsilon$ for all $i, j \in \overline{1,n}$.

The least common multiple of the orders of all permutations of the i-th inversion system is called *the order of the i-th inversion system.*

The least common multiple of the orders of all inversion systems is called *the order of an inversion matrix.*

An element $e \in Q$ is called an identity for an n-quasigroup $Q(\,)$ if $(x, e, e) = (e, x, e) = (e, e, x) = x$ for any $x \in Q$. *An n-loop* is an n-quasigroup with an identity.

The permutations I_{ij} on a set Q for an n-IP-loop with an identity e are defined as follows: $(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}) = e$ for any $x \in Q$ and any $i, j \in \overline{1, n}$ [1]. The matrix $[I_{ij}]_e$ for a 3-IP-loop has the form:

$$[I_{ij}] = \begin{bmatrix} \varepsilon & I_{12} & I_{13} & \varepsilon \\ I_{21} & \varepsilon & I_{23} & \varepsilon \\ I_{31} & I_{32} & \varepsilon & \varepsilon \end{bmatrix}.$$

If $(\varepsilon, \upsilon_{i2}, \upsilon_{i3}, \varepsilon)$ is the i-th inversion system of a 3-IP-loop, $i \in \overline{1, 3}$, then $(\varepsilon, \upsilon_{i2}^{2n-1}, \upsilon_{i3}^{2n-1}, \varepsilon)$ is also the i-th inversion system of some inversion matrix of this 3-IP-loop, and $(\varepsilon, \upsilon_{i2}^{2n}, \upsilon_{i3}^{2n}, \varepsilon)$ is an autotopy of this 3-IP-loop [2]. The main definitions and results for a 3-IP-loop are true for a 4-IP-loop as well.

3 Permutations of inversion systems and matrices of 3-IP-loops

The obtained results relative to nonsymmetric 3-IP-loops (to non-3-TS-loops).

Proposition 1. *If $Q()$ is a 3-IP-loop with an inversion matrix $[\upsilon_{ij}]$, $i \in \overline{1, 3}$, $j \in \overline{1, 4}$ and with an identity e, then any non-identity inversion permutation in even power of any inversion system does not leave fixed the identity e.*

Corollary 1. *If $Q()$ is a 3-IP-loop with an inversion matrix $[\upsilon_{ij}]$, $i \in \overline{1, 3}$, $j \in \overline{1, 4}$, and with an identity e, then any non-identity inversion permutation of any inversion system only in odd power leaves fixed the identity e.*

It means that for any non-identity inversion permutation υ_{ij}, $i, j \in \overline{1, 3}$, of an inversion matrix of a 3-IP-loop there exists odd number $2n + 1, n \in N$, such that $\upsilon_{ij}^{2n+1} e = e$, i.e., the identity of a loop is in a cycle of odd length in this inversion permutation.

Theorem 1. *The matrix $[I_{ij}]_e$ is an inversion matrix for any 3-IP-loop with an identity e.*

4 Permutations of inversion systems and matrices of 4-IP-loops

The obtained results relative to 4-IP-loops that are not 4-TS-loops and, in contrast to the ternary case. Another approach is required for the proof of analogous results.

Proposition 2. *If $Q(\,)$ is a 4-IP-loop with an inversion matrix $[\upsilon_{ij}]$, $i \in \overline{1,4}$, $j \in \overline{1,5}$ and with an identity e, then any non-identity inversion permutation in even power of any inversion system does not leave fixed the identity e.*

Corollary 2. *If $Q(\,)$ is a 4-IP-loop with an inversion matrix υ_{ij}, $i \in \overline{1,4}$, $j \in \overline{1,5}$, and with an identity e, then any non-identity inversion permutation of any inversion system only in odd power leaves fixed the identity e.*

These results are used in the proof of the following

Theorem 2. *The matrix $[I_{ij}]_e$ is an inversion matrix for any 4-IP-loop with an identity e.*

The question about existence of the matrix $[I_{ij}]_e$ for any n-IP-loop, $n > 4$, is still opened.

References

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