About one special inversion matrix of non-symmetric n-IP-loop

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Abstract

It is known that n-IP-quasigroups have more than one inversion matrix [1]. It is proved that one of these inversion matrices in the class of non-symmetric n-IP-loops is so-called matrix $[I_{ij}]$ of permutations, any of which has order two and fixes the unit element of the loop.

Keywords: quasigroup, loop, n-IP-quasigroup, n-IP-loop, inversion permutation, inversion matrix, isostrophism.

1 Main concepts and definitions

A quasigroup $Q(A)$ of arity n, $n \geq 2$, is called an n-IP-quasigroup if there exist permutations ν_{ij}, $i, j \in \overline{1,n}$ of the set Q, such that the following identities are true:

$$A(\{\nu_{ij}x_j\}_{j=1}^{i-1}, A(x_1^n), \{\nu_{ij}x_j\}_{j=i+1}^{n}) = x_i, \tag{1}$$

for all $x_1^n \in Q^n$, where $\nu_{ii} = \nu_{i\,n+1} = \varepsilon$. Here ε denotes the identity permutation of the set Q [1]. See [1] for more information on n-ary quasigroups.

The matrix

$$\left[\nu_{ij}\right] = \begin{bmatrix} \varepsilon & \nu_{12} & \nu_{13} & \dots & \nu_{1n} & \varepsilon \\ \nu_{21} & \varepsilon & \nu_{23} & \dots & \nu_{2n} & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \nu_{n1} & \nu_{n2} & \nu_{n3} & \dots & \varepsilon & \varepsilon \end{bmatrix}$$

is called *an inversion matrix* for a n-IP-quasigroup, the permutations $\nu_{i,j}$ are called *inversion permutations*. Any i-th row of an inversion matrix is called *i-th inversion system* for a n-IP-quasigroup.

The least common multiple (LCM) of orders of permutations of i-th inversion system is called the order of this system. The least common multiple (LCM) of orders of all inversion systems is called the order of inversion matrix.

The operation

$$B(x_1^n) = \alpha_{n+1}^{-1} A(\alpha_1 x_1, \ldots, \alpha_n x_n),$$

for all $x_1^n \in Q$, where α_1^{n+1} are permutations of the set Q, is called an isotope of the n-ary quasigroup $Q(A)$. If $A = B$, then we have an autotopy of the n-ary quasigroup $Q(A)$.

Recall that an n-ary quasigroup is an n-ary groupoid $Q(A)$, such that in the equality $A(x_1, x_2, \ldots, x_n) = x_{n+1}$ any n elements of the set $\{x_1, x_2, \ldots, x_n, x_{n+1}\}$ uniquely specifies the remaining one [1]. Therefore we can define a new quasigroup operation

$$^{\pi_i}A(x_1^{i-1}, x_{n+1}, x_{i+1}^n) = x_i, \tag{2}$$

that is called *the i-th inverse operation of the operation A.*

Let σ be a permutation of a set that consists from $(n+1)$ elements. The operation

$$^{\sigma}A(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)}$$

is called the σ-parastrophe of the operation A. If $\sigma(n+1) = n+1$, then we call this parastrophe a main parastrophe.

Isostrophy is a combination of an isotopy T and a parastrophy σ, i.e., an isostrophic image of an n-ary quasigroup $Q(A)$ is a parastrophic image of its isotopic image, and it is denoted by $A^{(\sigma,T)}$. If $A^{(\sigma,T)} = A$, then the pair (σ, T) is called an autostrophy of the n-ary quasigroup $Q(A)$ [1].

From identity (1) it follows that, for n-ary-IP-quasigroup $Q(A)$, the expression $T_i^2 = (\varepsilon, \nu_{i2}^2, \nu_{i3}^2, \ldots, \nu_{ii-1}^2, \varepsilon, \nu_{ii+1}^2, \nu_{in}^2, \varepsilon)$ is an autotopy of the n-ary quasigroup $Q(A)$.

Therefore
$$^{\pi_i}A = A^{T_i} \tag{3}$$
and
$$A^{(\pi_i,T_i)} = A \tag{4}$$
for all $i \in \overline{1,n}$. Any of equalities (3) and (4) defines an n-IP-quasigroup.

Below, for convenience, we denote the operation A by $()$.

An element e is called a unit of the n-ary operation $Q()$, if the following equality is true: $(\overset{i-1}{e}, x, \overset{n-i}{e}) = x$, for all $x \in Q$ and $i \in \overline{1,n}$. n-Ary quasigroups with unit elements are called n-ary loops [1, 2]. Loops of arity $n > 2$ can have more than one unit element [1]. n-IP-quasigroups with an least one unit element are called n-IP-loops [2, 3].

Permutations I_{ij} of the set Q are defined by the equalities
$$(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}) = e,$$
for all $x \in Q$ and $i, j \in \overline{1,n}$.

If the tuple $(\varepsilon, \nu_{12}, \nu_{13}, \dots, \nu_{1n}, \varepsilon)$ is the first inversion system of n-IP-quasigroup $Q()$, with the inversion matrix $[\nu_{ij}]$, then the tuple
$$(\varepsilon, \nu_{12}^{2n-1}, \nu_{13}^{2n-1}, \dots, \nu_{1n}^{2n-1}, \varepsilon),$$
is also an (first) inversion system, since the tuple $(\varepsilon, \nu_{12}^{2n}, \nu_{13}^{2n}, \dots, \nu_{1n}^{2n}, \varepsilon)$ is an autotopy of the quasigroup $Q()$. This is true for other ($i = 2, 3, \dots$) inversion systems.

Consider the matrix
$$[I_{ij}] = \begin{bmatrix} \varepsilon & I_{12} & I_{13} & \dots & I_{1n} & \varepsilon \\ I_{21} & \varepsilon & I_{23} & \dots & I_{2n} & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ I_{n1} & I_{n2} & I_{n3} & \dots & \varepsilon & \varepsilon \end{bmatrix}$$

An n-Quasigroup $Q(A)$ is called symmetric, if $A(x_{\varphi 1}^{\varphi n}) = A(x_1^n)$, for all $\varphi \in S_n$, where S_n is the symmetric group defined on the set Q, otherwise it is called non-symmetric [2, 3].

2 Main results

The first constructed example of an 3-IP-loop have the inversion matrix $[I_{ij}]$. V.D. Belousov proposed the following problem: is it true that any n-IP-loop has among inversion matrices the matrix $[I_{ij}]$?

Lemma. *If $Q()$ is a non-symmetric n-IP-loop with the inversion matrix $[\nu_{ij}]$ and unit e, then any non-identity inversion permutation from any inversion matrix of even order does not fix the unit element e.*

Corollary. *If $Q()$ is a non-symmetric n-IP-loop with the inversion matrix $[\nu_{ij}]$ and unit e, then any non-identity inversion permutation from any inversion matrix of odd order fix the unit element e.*

Theorem. *The matrix $[I_{ij}]$ is one of the inversion matrices in any non-symmetric n-IP-loop.*

References

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