Center conditions for a cubic system with two invariant straight lines and one invariant cubic

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We consider the cubic differential system of the form

$$
\begin{aligned}
\dot{x} &= y + ax^2 + cxy - y^2 + [(a-1)(p+c-b)+g]x^3 + \\
&\quad + [(b-p)(p+c-b) - a - n - 1]x^2y + pxy^2, \\
\dot{y} &= -x - gx^2 - dxy - by^2 + (a-1)(d+n-1)x^3 + \\
&\quad + [(b-p)(d+n+1) - g]x^2y + nxy^2 + by^3,
\end{aligned}
\tag{1}
$$

where the variables $x = x(t), y = y(t)$ and coefficients a, b, c, d, g, p, n are assumed to be real. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a fine focus.

It is easy to verify that the cubic system (1) has two invariant straight lines of the form $l_1 \equiv 1 + A_1x - y = 0,\ \ l_2 \equiv 1 + A_2x - y = 0$, where A_1, A_2 are distinct solutions of the equations $A^2 + (b-c-p)A - d - n - 1 = 0$, and we determine the conditions under which the cubic system (1) has also one irreducible invariant cubic curve of the form

$$\Phi(x,y) \equiv x^2 + y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 = 0$$

with $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$ and $a_{30}, a_{21}, a_{12}, a_{03} \in \mathbb{R}$. The problem of the center for cubic system (1) with: two parallel invariant straight lines and one invariant cubic $\Phi = 0$ was solved in [1]; a bundle of three algebraic curves $l_1 = 0$, $l_2 = 0$ and $\Phi = 0$ was solved in [2].

In this paper we study the problem of the center for cubic system (1) having three algebraic solutions $l_1 = 0,\ l_2 = 0,\ \Phi = 0$ in generic position and prove the following theorem:

Theorem 1. *Let the cubic system (1) have two invariant straight lines $l_1 = 0,\ l_2 = 0$ and one irreducible invariant cubic $\Phi = 0$. Then a fine focus $O(0,0)$ is a center if and only if the first three Lyapunov quantities vanish.*

Bibliography

[1] D. Cozma, *The problem of the center for cubic systems with two parallel invariant straight lines and one invariant cubic*, ROMAI Journal, 2015, no. 2, 63–75.

[2] D. Cozma, A. Dascalescu, *Integrability conditions for a class of cubic differential systems with a bundle of two invariant straight lines and one invariant cubic*, Bull. Acad. Sci. of Moldova, Mathematics, 2018, no.1, 120–138.