Invariant conditions of stability of unperturbed motion described by cubic differential system with quadratic part of Darboux type

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In [1] the center-affine invariant conditions of stability of unperturbed motion, described by critical two-dimensional differential systems with quadratic nonlinearities $s(1,2)$, cubic nonlinearities $s(1,3)$ and fourth-order nonlinearities $s(1,4)$, were obtained.
We consider the two-dimensional cubic differential system $s(1,2,3)$ of perturbed motion of the form

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \equiv \sum_{i=1}^{3} P^j_i,\ (j,\alpha,\beta,\gamma = 1,2), \tag{1}$$

where coefficients $a^j_{\alpha\beta}$ and $a^j_{\alpha\beta\gamma}$ are symmetric tensors in lower indices in which the total convolution is done. Coefficients and variables in (1) are given over the field of real numbers.
Let φ and ψ be homogeneous comitants of degree ρ_1 and ρ_2 respectively of the phase variables

$x = x^1$ and $y = x^2$ of a two-dimensional polynomial differential system. Then by [2] the transvectant

$$(\varphi, \psi)^{(j)} = \frac{(\rho_1 - j)(\rho_2 - j)}{\rho_1!\rho_1!} \sum_{i=0}^{j} (-1)^j \begin{pmatrix} j \\ i \end{pmatrix} \frac{\partial^j \varphi}{\partial x^{j-i} \partial y^i} \frac{\partial^j \psi}{\partial x^i \partial y^{j-i}} \qquad (2)$$

is also a comitant for this system.
In the works of Iurie Calin, see for example [3], it is shown that by means of the transvectant (2) all generators of the Sibirsky algebras of comitants and invariants for any system of type (1) can be constructed.
According to [3], we write the following comitants of the system (1):

$$R_i = P_i^1 x^2 - P_i^2 x^1, \quad S_i = \frac{1}{i}\left(\frac{\partial P_i^1}{\partial x^1} + \frac{\partial P_i^2}{\partial x^2}\right), \ i = 1, 2, 3. \qquad (3)$$

Later on, we will need the following comitants and invariants of system (1), built by operations (2) and (3), presented by Iurie Calin:

$$I_1 = S_1, \quad I_2 = (R_1, R_1)^{(2)}, \quad I_3 = \left((R_3, R_1)^{(2)}, R_1\right)^{(2)}, \ I_4 = (S_3, R_1)^{(2)},$$

$$K_2 = R_1, \quad K_5 = S_2, \quad K_8 = R_3, \quad K_9 = (R_3, R_1)^{(1)}, \quad K_{10} = (R_3, R_1)^{(2)},$$

$$K_{11} = \left((R_3, R_1)^{(2)}, R_1\right)^{(1)}, K_{14} = (S_2, R_1)^{(1)}, K_{15} = S_3, K_{16} = (S_3, R_1)^{(1)}.$$

Let for system (1) the invariant conditions $I_1^2 - I_2 = 0$, $I_1 < 0$ are hold. Then the system (1) becomes critical system of Lyapunov type [1].
Let for system (1) the invariant condition $R_2 \equiv 0$ is hold. Then the quadratic part of this system takes the Darboux form: $P_2^1 = x^1(a_{11}^1 x^1 + 2a_{12}^1 x^2)$, $P_2^2 = x^2(a_{11}^1 x^1 + 2a_{12}^1 x^2)$.

Theorem. *Let for system of perturbed motion* (1) *the invariant conditions* $I_1^2 - I_2 = 0$, $I_1 < 0$ *and* $R_2 \equiv 0$ *be satisfied. Then the stability of the unperturbed motion is described by one of the following twelve possible cases:*

I. $\mathcal{N}_1 \not\equiv 0$, *then the unperturbed motion is unstable;*
II. $\mathcal{N}_1 \equiv 0$, $\mathcal{N}_2 > 0$, *then the unperturbed motion is stable;*
III. $\mathcal{N}_1 \equiv 0$, $\mathcal{N}_2 < 0$, *then the unperturbed motion is unstable;*
IV. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv 0$, $K_5\mathcal{N}_3 \not\equiv 0$, *then the unperturbed motion is unstable;*
V. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv K_5 \equiv 0$, $\mathcal{N}_3\mathcal{N}_4 > 0$, *then the unperturbed motion is unstable;*
VI. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv K_5 \equiv 0$, $\mathcal{N}_3\mathcal{N}_4 < 0$, *then the unperturbed motion is stable;*
VII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv K_5 \equiv 0$, $\mathcal{N}_3 \not\equiv 0$, $\mathcal{N}_5 > 0$, *then the unperturbed motion is stable;*
VIII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv K_5 \equiv 0$, $\mathcal{N}_3 \not\equiv 0$, $\mathcal{N}_5 < 0$, *then the unperturbed motion is unstable;*
IX. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv K_5 \equiv 0$, $S\mathcal{N}_3 > 0$, *then the unperturbed motion is unstable;*
X. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv K_5 \equiv 0$, $S\mathcal{N}_3 < 0$, *then the unperturbed motion is stable;*
XI. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv 0$, $\mathcal{N}_3 \equiv 0$, *then the unperturbed motion is stable;*
XII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv \mathcal{N}_5 \equiv K_5 \equiv S \equiv 0$, *then the unperturbed motion is stable;*

where $\mathcal{N}_1 = 2K_{14} - I_1 K_5$, $\mathcal{N}_2 = 2I_1^2 K_{10} - 4I_1 K_{11} - 3I_1 I_2 K_{15} - 3I_1^2 K_{16} + 4I_3 K_2 + 3I_1 I_4 K_2$, $\mathcal{N}_3 = -12I_1 K_{10} K_2 + 8K_{11} K_2 + 3I_1^2 K_{15} K_2 - 6I_1 K_{16} K_2 + 6I_4 K_2^2 - 4I_1^3 K_8 + 8I_1^2 K_9$, $\mathcal{N}_4 = 2I_3 + I_1 I_4$, $\mathcal{N}_5 = 2K_{10} + I_1 K_{15} - K_{16}$, $S = 3K_{15} K_2 - 2I_1 K_8 - 4K_9$.

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, and all motions, which are sufficiently close to unperturbed motion, will be stable. In this case for sufficiently small perturbations any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series.

Bibliography

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