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Methods of solving of the optimal stabilization problem for stationary smooth control systems Part II Ending

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Abstract

In this article some ideas of Hamilton mechanics and differential-algebraic Geometry are used to exact definition of the potential function (Bellman-Lyapunov function) in the optimal stabilization problem of smooth finite-dimensional systems

3 Geometrical methods. Continuation

In the given section the greater attention is given to invariant description of the potential function and its Lagrange manifold.

3.4 Isomorphisms of algebras of smooth functions with marked derivation

Let's say, that the vector field $X = \xi^i(x)\frac{\partial}{partialx^i}$ is in the general position in relation to function $V(x)$, if $(n-1)$-th Lie derivative $L_X^{(n-1)}(V) = X^{(n-1)}(V) \neq F(V)$. Vector fields X, being in the general position to the function V, will form an open dense set in space of jets of smooth cuts $J^p(R^n,TR^n)$, $p = 0,1,2,\ldots,\infty$ of tangent stratification TR^n in topology of pointwise convergence. Therefore choice of an appropriate vector field is rather free. However the set of vector field of the general position in relation to the function $V(x)$ is not transitive concerning automorphisms $TR^n \rightleftarrows TR^n$, saving function $V(x)$. System of attributes defining orbits of an appropriate operation, can be the set of singularities of a vector field position X relative to level surfaces of functions $V(x)$, $X^{(1)}(V(x)), \ldots, X^{(n-2)}(V(x))$ along some submanifolds (tangency points).

Apparently, open orbit is the set of vector field of the general position to $V(x)$, defining orientation of foliation $V(x) = const$.

Obviously, the isomorphism of one (free) algebra of smooth functions with the marked element and derivation of the general position in relation to this element in other similar one is uniquely determined, if it exists, to within a discrete subgroup of automorphisms of algebra. In this case transfer of the differential equations $\Phi_{\underline{j}}(X^{(n+\underline{j})}(V), X^{(n-1)}(V), \ldots, V) = 0$, $\underline{j} = 0, 1, 2, \ldots$ takes place, whom the element V onto its pattern V' (with replacement X on the pattern X') satisfies. At a choice instead of X' other vector field of the general position X'' concerning V' equations $\Phi_{\underline{j}}(X'^{n+\underline{j}}(V'), X'^{n-1}(V'), \ldots, V') = 0$, $\underline{j} = 0, 1, 2, \ldots$ and equations $\Phi_{\underline{j}}(X''^{n+\underline{j}}(V'), X''^{n-1}(V'), \ldots, V') = 0$, $\underline{j} = 0, 1, 2, \ldots$ determine the same ideal in algebra of smooth functions on space of a stratification of jets of mappings $R^n \to R$. As derivation of the general position X concerning $V(x)$ it is possible to offer $X = C^{ji} V_j \frac{\partial}{\partial x^i}$, where C^{ji} – symmetrical positive definite matrix of constants, for which $C^{ji} V_j V_l \neq F(V)$, for example, can correspond $C^{ji} = \delta^{ji} + 1$.

Example 5.

For canonical nondegenerate potential function of two variables $V(x) = (x^1)^2 + (x^2)^2$ and vector field

$$X = (2V_1 + V_2)\frac{\partial}{\partial x^1} + (V_1 + 2V_2)\frac{\partial}{\partial x^2} = (4x^1 + 2x^2)\frac{\partial}{\partial x^1} + (2x^1 + 4x^2)\frac{\partial}{\partial x^2}$$

it is possible to obtain the relation $V^{(2)} = 16(V^{(1)} - 3V)$ between function and its two derivatives of the first order $V^{(1)}$, $V^{(2)}$ along X, which holds for each element of some (open) set of uncanonical nondegenerate potential functions and appropriate vector field $X = (2V_1 + V_2)\frac{\partial}{\partial x^1} + (V_1 + 2V_2)\frac{\partial}{\partial x^2}$. Therefore in the problem of the optimum stabilization of a system of the second order with a nondegenerate Bellman-Lyapunov function the following relation can be used additionally

$$(2V_1+V_2)^2V_{11}+(4(V_1)^2+10V_1V_2+4(V_2)^2)V_{12}+(V_1+2V_2)^2V_{22}=$$

$$=8(2(V_1)^2+2V_1V_2+2(V_2)^2-3V).$$

3.5 Representation of the first integrals and evaluation method of Lagrangian manifold

If ψ – is the first integral of the Hamilton system (2.1), connected with the problem of optimal stabilisation, then $\psi = c\ mod(I(L^+)) = c\,mod(I(L^-))$, where $c-const$, $I(L^+)$, $I(L^-)$ – ideals of manifolds L^+, L^-; that is $\psi_0 \equiv (\psi - c) \in I(L^+)\bigcap I(L^-)$. Let $\{a_i \equiv V_i - \alpha_{ij}(x)x^j\}_{i=1}^n$ – independent generators $I(L^+)$; $\{b_i \equiv V_i - \beta_{ij}(x)x^j\}_{i=1}^n$ – independent generators $I(L^-)$; $a_i \notin I(L^-)$, $b_j \notin I(L^+)$.

We have $I(L^+)\bigcap I(L^-) = I(L^+)\cdot I(L^-)$ (the intersection of ideals coincides with the product). Really, $I(L^+)\cdot I(L^-) \subset I(L^+)\bigcap I(L^-)$ (always); back: let $r = s^i a_i = t^j b_j$; s^i, $t^j \in C^\infty(T^*R^n)$; common solution of last equation concerning s^i, t^j:

$$s^i = A^{ik}a_k + B^{ik}b_k, \quad -t^j = C^{jl}a_l + D^{jl}b_l;$$

$$A^{ik} = -A^{ki},\ D^{jl} = -D^{lj},\ C^{jl} = -B^{lj} \in C^\infty(T^*R^n)\ ;$$

$$r = s^i a_i = A^{ik}a_k a_i + B^{ik}b_k a_i = B^{ik}b_k a_i \in (I(L^+)\cdot I(L^-)),$$

i.e. $I(L^+)\bigcap I(L^-) \subset I(L^+)\cdot I(L^-)$.

Thus each integral ψ_0 (including φ), that becomes equal to zero in the origin of coordinates $0 \in T^*R^n$, belongs to the product $I(L^+)\cdot I(L^-)$.

Thus:

— ideals $I(L^+)$, $I(L^-)$ are invariant concerning derivation $Id\varphi$;

— generators a_i, b_j are not the first integrals of the Hamilton system (2.1).

Therefore at presence of some basis elements a_α or b_β, α, β run through the proper subset $\{1,\dots,n\}$ the generation of full family of generators of ideals $I(L^+)$, $I(L^-)$ is possible by application of derivation $Id\varphi$:

$$a_\alpha^{(1)} \equiv (Id\varphi)(a_\alpha) \quad [b_\beta^{(1)} \equiv (Id\varphi)(b_\beta)]$$

$$a_\alpha^{(2)} \equiv (Id\varphi)^2(a_\alpha) \quad [b_\beta^{(2)} \equiv (Id\varphi)^2(b_\beta)] \quad \text{etc.}$$

The following calculation algorithm L^+ can be used:
— $a_1 = V_1 - \alpha_{1j}(x)x^j$ is selected, where $\alpha_{1j}(x)$ – undefined elements $C^\infty(R^n)$;
— the $a_1^{(1)}, \dots, a_1^{(n)}, \dots$ are found in.

There exists the implication

$$(\ a_1 = a_1^{(1)} = ... = a_1^{(n-1)} = 0\ ) \Rightarrow (\ a_1^{(\delta)} = 0,\ \delta = n, n+1, ...\ ),$$

from which follows a system of differential partial equations concerning α_{1j}. If the conditions of an integrability $V_{ix^j} = V_{jx^i}$, deduced from $a_1 = a_1^{(1)} = \ldots = a_1^{(n-1)} = 0$ would be added to the obtained system, the resulting system completely will describe $L^+ \bigcup L^-$. For extraction L^+ it is necessary to take into account positive determinancy of the matrix $(V_{ix^j})_{i,j=1}^n$.

In case of linearly-square problem α_{1j} – *const*. Therefore here the number of unknowns from n^2 (Riccaty system) decreases to n.

Example 6.

$$\begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = x^3 \\ \dot{x}^3 = u \end{cases} \qquad J = \int_0^\infty ((x^1)^2 + u^2)dt$$

$u_{opt} = -\frac{1}{2}V_3$ – optimal control on L^+
$\varphi = x^2V_1 + x^3V_2 - \frac{1}{4}(V_3)^2 + (x^1)^2$ – Hamiltonian;
$Id\varphi = x^2\frac{\partial}{\partial x^1} + x^3\frac{\partial}{\partial x^2} - \frac{1}{2}V_3\frac{\partial}{\partial x^3} - 2x^1\frac{\partial}{\partial V_1} - V_1\frac{\partial}{\partial V_2} - V_2\frac{\partial}{\partial V_3}$ – Hamilton vector field.
Let $a_1 \equiv (V_1 - \alpha x^1 - \beta x^2 - \gamma x^3) \in I(L^+)$, $\alpha,\ \beta,\ \gamma$ – *const* ;
$a_2 \equiv Id\varphi(a_1) = (-\alpha x^2 - \beta x^3 + \frac{1}{2}\gamma V_3 - 2x^1) \in I(L^+)$;
$a_3 \equiv (Id\varphi)^2(a_1) = (-\alpha x^3 + \frac{1}{2}\beta V_3 - \frac{1}{2}\gamma V_2 - 2x^2) \in I(L^+)$;

$a_4 \equiv (Id\varphi)^3(a_1) = (\frac{1}{2}\alpha V_3 - \frac{1}{2}\beta V_2 + \frac{1}{2}\gamma V_1 - 2x^3) \in I(L^+)$.

Solving jointly system $a_1 = a_2 = a_3 = a_4 = 0$, taking into account symmetry and positive determinancy of the matrix of coefficients of expansion of variables V_i through x^j, we obtain missing parameters of the ideal $I(L^+)$: $\alpha = 4$, $\beta = 4$, $\gamma = 2$, for which

$$\begin{cases} V_1 = 4x^1 + 4x^2 + 2x^3 \\ V_2 = 4x^1 + 6x^2 + 4x^3 \\ V_3 = 2x^1 + 4x^2 + 4x^3 \end{cases} \qquad - \text{ equations } \mathrm{L}^+ .$$

Example 7 [14, page 51].

$$\begin{cases} \dot{y}^1 = y^2 \\ \dot{y}^2 = y^3 \\ \dot{y}^3 = y^2 \cos y^1 + u \end{cases}$$

$J = \int\limits_0^\infty ((y^1 + y^2 + y^3)^2 + (y^2 + y^3 + (y^2 \cos y^1 + u))^2)dt$

$u_{opt} = -\frac{1}{2}V_3 - y^2 - y^3 - y^2 \cos y^1$ – optimal control on L^+;

$\varphi = y^2 V_1 + y^3 V_2 - (y^2 + y^3)V_3 - \frac{1}{4}(V_3)^2 + (y^1 + y^2 + y^3)^2$ – Hamiltonian;

$Id\varphi = y^2 \frac{\partial}{\partial y^1} + y^3 \frac{\partial}{\partial y^2} - (y^2 + y^3 + \frac{1}{2}V_3)\frac{\partial}{\partial y^3} - 2(y^1 + y^2 + y^3)\frac{\partial}{\partial V_1} -$
$-(V_1 - V_3 + 2y^1 + 2y^2 + 2y^3)\frac{\partial}{\partial V_2} - (V_2 - V_3 + 2y^1 + 2y^2 + 2y^3)\frac{\partial}{\partial V_3}$ – Hamilton vector field.

$a_1 \equiv V_1 - ay^1 - by^2 - cy^3$;

$a_1^{(1)} \equiv (Id\varphi)(a_1) = -2y^1 + (-2 - a + c)y^2 + (-2 - b + c)y^3 + \frac{1}{2}cV_3$;

$a_1^{(2)} \equiv (Id\varphi)^2(a_1) = -cy^1 + (b - 2c)y^2 + (b - a - c)y^3 - \frac{1}{2}cV_2 + (1 + \frac{1}{2}b)V_3$;

$a_1^{(3)} \equiv (Id\varphi)^3(a_1) = (c - b - 2)y^1 + (a + c - 2 - 2b)y^2 + (a - 2 - b)y^3 +$
$+ \frac{1}{2}cV_1 - (1 + \frac{1}{2}b)V_2 + (\frac{1}{2}a + 1)V_3$.

Solving jointly system $a_1 = a_1^{(1)} = a_1^{(2)} = a_1^{(3)} = 0$, taking into account symmetry and positive determinancy of the matrix of coefficients of expansion of variables V_i y^j , we obtain missing parameters ofthe ideal $I(L^+)$: $a = 2,\ b = 2,\ c = 2$, for which

$$\begin{cases} V_1 = 2y^1 + 2y^2 + 2y^3 \\ V_2 = 2y^1 + 2y^2 + 2y^3 \\ V_3 = 2y^1 + 2y^2 + 2y^3 \end{cases} \qquad - \text{ equations } \mathrm{L}^+ .$$

$$u_{opt} = -y^1 - 2y^2 - 2y^3 - y^2 \cos y^1 .$$

4 Symmetry in the problem of the optimal stabilization

The possibility of effective definition of symmetries of the differential equations [45,46] is based that derivation $X^{(0)}$ in algebra of smooth functions $C^\infty(R^n \times R^r)$ on the space of stratification $\pi : R^n \times R^r \to R^n : (x^i, y^\alpha) \mapsto (x^i)$ uniquely proceeds up to derivation $X^{(q)}$, $q = 1, 2, \ldots, \infty$, algebras $C^\infty(J^{(q)}(R^n, R^r))$ of smooth functions on space of stratification $J^{(q)}(R^n, R^r) : \{x^i, y^\alpha, y^\alpha_j, ..., y^\alpha_{j_1 \ldots j_q}\}$
$q-$jets of cuts of the stratification π with preservatioin of the forms

$$\Delta^\alpha \equiv dy^\alpha - y^\alpha_i dx^i \ ;$$

$$\Delta^\alpha_i \equiv dy^\alpha_i - y^\alpha_{ij} dx^j \ ;$$

$$. \quad . \quad . \quad . \quad . \quad .$$

$$\Delta^\alpha_J \equiv dy^\alpha_J - y^\alpha_{Jk} dx^k \ ;$$

$$J = (j_1, \ldots, j_q) \ - \text{multiindex of length q (\#J = q);}$$

etc.

If

$$X^{(0)} \equiv \eta^\alpha(y^\beta, x^i) \frac{\partial}{\partial y^\alpha} + \xi^j(y^\beta, x^i) \frac{\partial}{\partial x^j},$$

then

$$X^{(q)} = X^{(0)} + \sum_{\substack{\alpha = 1, ..., r \\ \#J = 1, ..., q}} \eta^\alpha_J \frac{\partial}{\partial y^\alpha_J},$$

where

$$\eta^\alpha_J = D_J(\eta^\alpha - y^\alpha_k \xi^k) + y^\alpha_{Jk} \xi^k \ ,$$

$$D_J \equiv D_{x^{j_q}} \circ D_{x^{j_{q-1}}} \circ ... \circ D_{x^{j_1}} .$$

It makes sense to set derivation X preserving the forms Δ^α_J, at once in algebra $C^\infty(J^\infty(R^n, R^r))$, thus it is quite defined by the values $\eta^\alpha[y]$, $\xi^j[y] \in C^\infty(J^\infty(R^n, R^r))$ on coordinate basis functions y^α , x^i . The formulas of a continuation (on other coordinate functions) coincide with above mentioned. In this case derivation is called generalized, and at $\xi^i[y] = 0$ as evolutionary.

Derivation X, representing a symmetry of the differential equations system

$$F_a[y] = 0 \ , \ F_a[y] \in C^\infty(J^\infty(R^n, R^r)) \ ,$$

is determined from a condition of invariancy of the ideal generated by the continued system, that is

$$XF_a \equiv P_a^{Jb} D_J F_b$$

for some smooth functions $P_a^{Jb} \in C^\infty(J^\infty(R^n, R^r))$.

4.1 Evolutionary symmetries of the Hamilton-Jacoby equation

The benefit of evolutionary symmetries [45,46] of the Hamilton-Jacoby equation $\varphi = 0$ for solution of the optimum stabilization problem is connected to that the Bellman-Lyapunov function $V(x)$ is an "almost isolated" solution of this equation. Really Lagrangian manifold L^+, being unique separatrix of steady points of the Hamilton system (2.1), represents an isolated (in topology of pointwise convergence) cut of the stratification $T^*R^n \bigcap \varphi^{-1}(0)$ in any neighbourhood of the origin of coordinates $0 \in R^n$. Therefore intersection of a rather small neighbourhood of function $V(x)$ (in usual jet topology $J^q(R^n, R)$ for any $q = 0, 1, 2, \dots, \infty$) with set of solutions of the equation $\varphi = 0$ will represent a segment of straight line $V(x) + c$, c – parameter.

From here follows, that the value of evolutionary derivation saving the Hamilton-Jacoby equation in the point $V(x)$ can be only constant (or that $V(x)$ is a stationary point an appropriate evolutionary vector field of vectors to within trivial shifts $k\frac{\partial}{\partial V}$, k – $const$).

Let's show, that the symmetries of the equation $\varphi = 0$ of a kind $Q(x^i, V)\frac{\partial}{\partial V}$ in most cases are trivial, and $Q(x^i, V_j)\frac{\partial}{\partial V}$ determine the first integral $Q(x^i, V_j)$ of an appropriate Hamilton system. Let

$$X \equiv Q(x^i, V)\frac{\partial}{\partial V}; \quad X^{(1)} = X + (D_{x^j}Q)\frac{\partial}{\partial V_j} = Q\frac{\partial}{\partial V} + (Q_{x^j} + Q_V V_j)\frac{\partial}{\partial V_j}$$

The symmetry of the equation $\varphi = 0$ concerning derivation X means, that there is the function $P \in C^\infty(J^\infty(R^n, R))$, for which

$$X^{(1)}\varphi \equiv P\varphi$$

or

$$(Q_{x^j}\varphi_{V_j} + (Q_V - P)\varphi - Q_V\omega \equiv 0;$$

in common case, φ_{V_j}, φ, $\omega \in C^\infty(J^\infty(R^n, R))$ – independent functions; hence, $Q_{x^j} = (Q_V - P) = Q_V = 0$, i.e. Q – *const*.

Let

$$X \equiv Q(x^i, V_k)\frac{\partial}{\partial V}; \qquad X^{(1)} = X + (D_{x^i}Q)\frac{\partial}{\partial V_i}$$

Noting the condition of symmetry $\varphi = 0$, collecting similar terms and taking into account, that functions $\varphi_{V_i} \in C^\infty(J^\infty(R^n, R))$ in common case are independent, we shall come to the equation

$$\text{Poisson bracket } (\mathrm{Q}, \varphi) = 0, \quad \text{i.e. } \mathrm{Q}(\mathrm{X}^{\mathrm{i}}, \mathrm{V_j}) \ - \text{ first integral } \mathrm{Id}\varphi.$$

Evolutionary vector field $\varphi(x^i, V_j)\frac{\partial}{\partial V}$ is a symmetry of the Hamilton-Jacoby equation and besides leaves all of its solutions fixed.

4.2 Defining equations of finite-dimensional symmetries in an optimal stabilisation problem

It is natural to search for infinitesimal forming groups of symmetries of the system (1.4.a), (1.4.b) of a kind

$$X = \xi^k(x)\frac{\partial}{\partial x^k} + \eta^0(x)\frac{\partial}{\partial V} + \eta^k(x, u)\frac{\partial}{\partial u^k} + (\eta^0_{x^i} - V_k\xi^k_{x^i})\frac{\partial}{\partial V_i}.$$

That is

$$\begin{cases}[(f^i_{x^k}\xi^k + f^i_{u^l}\eta^l)V_i + f^i(\eta^0_{x^i} - V_k\xi^k_{x^i}) + \omega_{x^k}\xi^k + \omega_{u^l}\eta^l]\Big|_{(1.4)} = 0\\ [(f^i_{u^jx^k}\xi^k + f^i_{u^ju^l}\eta^l)V_i + f^i_{u^j}(\eta^0_{x^i} - V_k\xi^k_{x^i}) + \omega_{u^jx^k}\xi^k + \omega_{u^ju^l}\eta^l]\Big|_{(1.4)} = 0\end{cases}$$

If $m < n$, then $m+1$ variables V_A, $A \in \{1, ..., m+1\}$ can be expressed from the system(1.4)

$$V_A = G^B_A(u,x)V_B + H_A(u,x), \quad A \in \{1, ..., m+1\}, \quad B \in \{m+2, ..., n\}.$$

Let's obtain defining equations of infinitesimal generators of symmetries group of the system (1.4)

$$\begin{cases}(f^A_{x^k}\xi^k + f^A_{u^l}\eta^l)G^B_A + (f^B_{x^k}\xi^k + f^B_{u^l}\eta^l) - f^i(\xi^A_{x^i}G^B_A + \xi^B_{x^i}) = 0\\ (f^A_{x^k}\xi^k + f^A_{u^l}\eta^l)H_A + f^i(\eta^0_{x^i} - H_A\xi^A_{x^i}) + \omega_{x^k}\xi^k + \omega_{u^l}\eta^l = 0\\ (f^A_{u^jx^k}\xi^k + f^A_{u^ju^l}\eta^l)G^B_A + (f^B_{u^jx^k}\xi^k + f^B_{u^ju^l}\eta^l) - f^i_{u^j}(\xi^A_{x^i}G^B_A + \xi^B_{x^i}) = 0\\ (f^A_{u^jx^k}\xi^k + f^A_{u^ju^l}\eta^l)H_A + f^i_{u^j}(\eta^0_{x^i} - H_A\xi^A_{x^i}) + \omega_{u^jx^k}\xi^k + \omega_{u^ju^l}\eta^l = 0\end{cases}$$

A specific interest is to investigate symmetries groups of optimal control systems of a special kind (for example, linearly-square concerning the control).

5 Additional methods

5.1 Synthesis of suboptimal control of a given structure

Let $L \subset \varphi^{-1}(0)$ – n-dimensional Lagrangian manifolds correctly projecting on $R^n\{x^i\}^n_{i=1}$ (with the generating function $V(x)$). Hamilton field $Id\varphi$ is tangent L. In this case a surjective homomorphism of algebras restriction, saving derivation $Id\varphi$, $C^\infty(T^*R^n) \to C^\infty(T^*R^n)/I(L)$, where $I(L)$ – ideal of the Lagrangian manifold L.

Algebra $C^\infty(T^*R^n)/I(L)$ is exactly isomorphic $C^\infty(R^n)$. Here x – projection $X = \varphi_{V_i}(x^j, V_k(x))\frac{\partial}{\partial x^i}$, where $V_k = V_k(x)$ – equations L, corresponds to derivation $Id\varphi$. Therefore there is surjective differential homomorphism $\rho : (C^\infty(T^*R^n), Id\varphi) \to (C^\infty R^n, X)$.

The indicated fact can be useful at an evaluation of suboptimal control of a given structure $u(x, \alpha)$, α – parameters. In this case $X \approx f^i(x, u(x, \alpha))\frac{\partial}{\partial x^i}$, f^i – right side of the system (1.1). Homomorphism ρ can not exist for all values $\alpha - const$, the following procedure nevertheless can be useful.

Comparing variables $x^i \in C^\infty(T^*R^n)$ with variables $x^i \in C^\infty(R^n)$, we differentiate them sufficient number of times, the first – along $Id\varphi$, the second – along X, so that to obtain $2n$ elements, (depending from α) in the first and second algebra. Eliminating variables x^i from the obtained elements in $C^\infty(R^n)$, we shall obtain n relations, defining "kernel" ρ. Let's express variables V_i through x and α (if not to make this, the rest part of the procedure should be changed). Let's select parameters α so that obtained "kernel" was "maximum holonomic", for it we minimize on α the function $\int_K[\sum_{i<j}(V_{ix^j} - V_{jx^i})^2]dx^1 \wedge \ldots \wedge dx^n$, K – is the selected compact neighbourhood of zero. From several solutions the parameters α are selected, for which the matrix $V_{ix^j}(x, \alpha)$ has positive main minors.

5.2 Functions continuing the Bellman-Lyapunov function on space of greater dimensionality

In some cases in the adjacent class $\rho^{-1}(V)$, where ρ: $(C^\infty(T^*R^n), Id\varphi) \to (C^\infty(R^n), X)$ – the differential homomorphism described in the previous item, function W exists, possessing the following positive properties:

$$L_{Id\varphi}(W) = -\omega;$$

$$W_{V_i} = 0 \text{ (and accordingly } \mathrm{W_{x^i} = V_i}) \ - \text{defining equations } \mathrm{L}^+.$$

As the equation $L_{Id\varphi}(W) = -\omega$ is linear, this mode can be useful.

However, the indicated property holds seldom. In more general case, it is possible to search as a solution of the Hamilton-Jacoby equation (1.5) function $W(x^i, y^\alpha)$, depending from additional variables y^α, $\alpha = 1, ..., r$, restriction of which on the surface $P : W_{y^\alpha}(x^i, y^\beta) = 0$, coincides with the Bellman-Lyapunov function $V(x^i)$. In this case

$W_{x^i}|_P = V_{x^i}$. The function $\varphi(x^i, W_{x^j}(x^k, y^\alpha))$ is contained in the ideal $(W_{y^1}, ..., W_{y^r})$.

It is possible to limit oneself by functions W of kind $W(x^i, y^\alpha) = F(x) + G_\alpha(x)y^\alpha + (y^1)^2 + ... + (y^r)^2$. For linearly-square systems $F(x)$ – homogeneous square, $G_\alpha(x)$ – homogeneous linear functions. From family of solutions a such one is selected, that $W|_P$ is positive definite.

Example 9.

$$\begin{cases} \dot{x}^1 = x^1 + x^2 \\ \dot{x}^2 = u \end{cases} \qquad\qquad J = \int\limits_0^\infty u^2 dt$$

$u_{opt} = -\frac{1}{2}V_2$ – optimal control on L^+;
$\varphi = (x^1 + x^2)V_1 - \frac{1}{4}(V_2)^2$ – Hamiltonian.
We search for function $W = a(x^1)^2 + b(x^2)^2 + \frac{1}{2}(x^3)^2 + dx^1x^2 + fx^1x^3 + kx^2x^3$, conversing to zero the Hamiltonian and satisfying the equation $W_{x^3} = 0$.
The appropriate relations for coefficients will be the following:
$4(2a - f^2) - (d - kf)^2 = 0$, $4(d - fk) - (2b - k^2)^2 = 0$, $4(2a - f^2) + 4(d - - fk) - 2(2b - k^2)(d - kf) = 0$.
At $a = 4$, $b = 4$, $d = 8$, $f = 2$, $k = 2$
the function $W = 4(x^1)^2 + 4(x^2)^2 + \frac{1}{2}(x^3)^2 + 8x^1x^2 + 2x^1x^3 + 2x^2x^3$,
surface $P:\ W_{x^3} = x^3 + 2x^1 + 2x^2 = 0$,
$W|_P = 2(x^1 + x^2)^2 \equiv V(x)$ – Bellman-Lyapunov function.

5.3 Rank conditions on the Lagrangian manifold

For a nondegenerate Bellman-Lyapunov function $V(x)$ there is a smooth function $k(x)$ ($k(x) > 0$ for all x from some neighbourhood of the origin of coordinates R^n), that $n-$form $(dV_1 \wedge ... \wedge dV_n - k(x)dx^1 \wedge ... \wedge dx^n)$, and also all forms, obtained from it by operations $i_{Id\varphi}$, d, are contained in a differential ideal, defining L^+.

For degenerate function $V(x)$ the first form of a sequence is substituted by $dV_1 \wedge ... \wedge dV_n$.

Example 10.

$$\begin{cases} \dot{x}^1 = -x^2 \\ \dot{x}^2 = (x^1)^2 + \sin x^1 + u \end{cases} \qquad J = \int_0^\infty ((x^1)^2 + \sin x^1 - x^2 + u)^2 dt$$

$u_{opt} = -\frac{1}{2}V_2 - (x^1)^2 - \sin x^1 + x^2$ – optimal control on L^+;
$\varphi = -x^2 V_1 + x^2 V_2 - \frac{1}{4}(V_2)^2$ – Hamiltonian;
$Id\varphi = -x^2 \frac{\partial}{\partial x^1} + (x^2 - \frac{1}{2}V_2)\frac{\partial}{\partial x^2} - (-V_1 + V_2)\frac{\partial}{\partial V_2}$ – Hamilton vector field.
Let us assume that $\beta^2 \equiv dV_1 \wedge dV_2 \in I(L^+)$,
then $\beta^1 = i_{Id\varphi}(\beta^2) = (V_2 - V_1)dV_1 \in I(L^+)$,
i.e. $V_1 = const = 0$ or $V_1 = V_2$.

In the first case from the Hamilton-Jacoby equation follows $V_1 = 0$, $V_2 = 4x^2$, in the second – $V_1 = V_2 = 0$. The Bellman-Lyapunov function corresponds to the first case $V(x) = 2(x^2)^2$.
$u_{opt} = -x^2 - (x^1)^2 - \sin x^1$.

Example 11.

$$\begin{cases} \dot{x}^1 = x^1 + x^2 \\ \dot{x}^2 = (x^1)^2 + \sin x^1 + u \end{cases} \qquad J = \int_0^\infty ((x^1)^2 + \sin x^1 + x^2 + u)^2 dt$$

$u_{opt} = -\frac{1}{2}V_2 - (x^1)^2 - \sin x^1 - x^2$ – optimal control on L^+;
$\varphi = (x^1 + x^2)V_1 - x^2 V_2 - \frac{1}{4}(V_2)^2$ – Hamiltonian;
$Id\varphi = (x^1 + x^2)\frac{\partial}{\partial x^1} + (-x^2 - \frac{1}{2}V_2)\frac{\partial}{\partial x^2} - V_1 \frac{\partial}{\partial V_1} - (V_1 - V_2)\frac{\partial}{\partial V_2}$ – Hamilton vector field.
Let us assume $\beta^2 \equiv dV_1 \wedge dV_2 \in I(L^+)$,
then $\beta^1 = i_{Id\varphi}(\beta^2) = -V_1 dV_2 + (V_1 - V_2)dV_1 \in I(L^+)$.
The form β^1 is closed ($d\beta^1 = 0$), $\int(\beta^1) = \frac{1}{2}((V_1)^2 - 2V_1V_2) + const$.
Possible variants of system solution

$$\begin{cases} \varphi = 0 \\ \frac{1}{2}((V_1)^2 - 2V_1V_2) = 0 \end{cases}$$

are the following

$V_1 = V_2 = 0$; $V_1 = 0$, $V_2 = -4x^2$;
$V_1 = 16x^1 + 8x^2$, $V_2 = 8x^1 + 4x^2$.
The last case describes L^+.
The Bellman-Lyapunov function $V(x) = 8(x^1)^2 + 2(x^2)^2 + 8x^1x^2$.
$u_{opt} = -3x^2 - 4x^1 - (x^1)^2 - \sin x^1$.

5.4 Exterior differential system

It is possible to make more precise setting of the problem of searching of the Bellman-Lyapunov function, by giving it in a canonical frame (for nondegenerate function $V = (t^1)^2 + ... + (t^n)^2$).

More completely this information can be taken into account, if we use the (closed) exterior differential system:

$$\begin{cases} \varphi(x^i, V_j) = 0 & \\ d\varphi(x^i, V_j) = 0 & dx^1 \wedge ... \wedge dx^n \neq 0 \\ dV_i \wedge dx^i = 0 & dt^1 \wedge ... \wedge dt^n \neq 0 \\ V_i dx^i = 2\delta_{jk} t^j dt^k & \end{cases} \tag{5.1}$$

As independent variables of the system (5.1) can be selected $\{x^i\}_{i=1}^n$ or $\{t^j\}_{j=1}^n$. At selected independent variables $\{x^i\}_{i=1}^n$ the solution $(V(x^i), t^1(x^i), ..., t^n(x^i))$ has uniquely defined first component at ambiguous remaining n ones. At independent variables $\{t^j\}_{j=1}^n$ the solution $(V(t^j), x^1(t^j), ..., x^n(t^j))$ – is ambiguous, however the function $V(x)$, obtained by elimination, t^j is uniquely defined. For selection of the unique solution of the system (5.1) it is necessary to impose additional conditions on functions of coordinate transformation $x^i = x^i(t^j)$. For example, in some correct set of a neighbourhood of the origin of coordinates not containing the origin of coordinates and such that intersection of any ray with the beginning $0 \in R^n$ with each surface of a positive level $V(x) = V > 0$ has only one point [such set with added zero $0 \in R^n$ contains a convex range of function definition $V(x)$ in a neighbourhood of zero] the function $V(x)$ together with $(t^1)^2 + ... + (t^n)^2$ lies on the same orbit (the local one) of group of radial stretch $\overline{x}^i = k(x)x^i$, where $k(x) > 0$ is in the correct set. Invariants (of order zero) of group of radial stretch are $\frac{x^\alpha}{x^1}$, $\alpha = 2, ..., n$. Therefore the equations

$$\frac{x^\alpha}{x^1} = \frac{t^\alpha}{t^1}, \quad \alpha = 2, ..., n \tag{5.1.a}$$

can be added to the system (5.1) for security of uniqueness of a solution in the correct set

6 Conclusion

In the work the various methods are represented which both individually and in a combination, can be useful at solving of the optimal stabilization problem of movement.

Many from these methods are non-traditional in the field of optimal control theory and admit further detaling and improvement.

The basic direction of further development of the given area can be connected with creation of the invariant theory of the pair "Hamiltonian- Lagrangian manifold" (φ, L) and separately with a theory of potential function V on the basis of invariant differential calculus.

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