CALCULATION ERRORS OF THE PLATES USING FINITE ELEMENT METHOD

S. Galbinean, Postgrad.
Technical University of Moldova

INTRODUCTION

The finite element method (FEM) is currently the most used numerical method of calculation It is very effective for studying various problems from different areas of engineering. FEM, usually leads to solving a system of algebraic equations with a large number of unknowns, so it is closely related to computer use. One of the basic problems of MEF is the difficulties of calculating the plates with different conditions on boundary. The problem is actual and were not proposed any solutions to solve it.

Below are studied rectangular plates with different ways of bearing edge and are estimated the errors occurring in their modeling by finite element method.

CALCULATION OF THE PLATE WITH DIFFERENT CONDITIONS ON BOUNDARY

To solve the problem for rectangular plates with various types of supports we use the solution proposed by L. Levy, considering the two opposite sides simply supported. If simply supported sides are $x = 0$ and $x = a$ (fig. 1), the deflection $w(x, y)$ can be expressed with relation

$$w(x, y) = \sum_{m=1}^{\infty} Y_m(y) \sin \frac{m\pi x}{a}. \qquad (1)$$

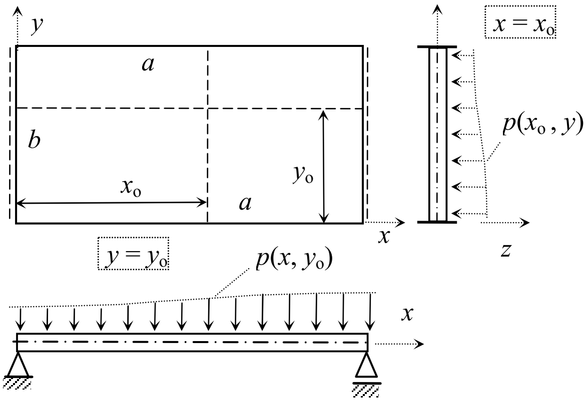

Figure 1. Plate having two opposite sides ($x = 0$; $x = a$) simply supported and the other two of any kind.

The load p(x,y) is presented in Fourier series of the same form

$$p(x, y) = \sum_m p_m(y) \sin \alpha_m x , \qquad (2)$$

where,

$$p_m(y) = \frac{2}{a} \int_0^a p(x, y) \sin(\alpha_m x)\, \mathrm{d}x , \qquad (3)$$

and

$$\alpha_m = m\pi / a . \qquad (4)$$

Frequently we meet two cases::

1) uniformly distributed load,

$$p_m(y) = \frac{4p}{m\pi} = const. \;, \; m = 1, 3, 5, ... \;; \qquad (5)$$

2) hydrostatic load distributed according to the law $p(x, y) = px/a$,

$$p_m(y) = \frac{2p}{m\pi}(-1)^{m+1} = const., \qquad (6)$$

$m = 1, 2, 3, ...$.

According to the classical theory of Kirchhoff's plates, the differential equation of deflection has the form

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}. \qquad (7)$$

Substituting relation (1) and (2) in the differential equation of the plate (7) we obtain:

$$Y_m'''' - 2\alpha_m^2 Y_m'' + \alpha_m^4 Y_m = p_m / D , \qquad (8)$$

which is a fourth order differential equation with constant coefficients. Noting with Y_m^{part} the particular solution of the equation (8) and considering that the roots $r_{1,\,2,\,3,\,4} = \pm\, \alpha_m\, y$ of its characteristic equation of the homogeneous equation are real, $r_{1,\,2,\,3,\,4} = \pm\, \alpha_m\, y$, the general solution will be presented in the form

$$Y_m(y) = Y_m^{\text{omog}} + Y_m^{\text{part.}} , \qquad (9)$$

where

$$Y_m(y) = A_m \cosh\alpha_m y + B_m \sinh\alpha_m y + \\ +C_m \alpha_m y\cosh\alpha_m y + D_m \alpha_m y \sinh\alpha_m y + Y_m^{\text{part.}} \quad (10)$$

Therefore the deflection becomes:

$$w(x,y) = \sum_{m=1}^{\infty}\left(A_m \cosh\alpha_m y + B_m \sinh\alpha_m y + \\ +C_m \alpha_m y\cosh\alpha_m y + D_m \alpha_m y \sinh\alpha_m y + Y_m^{\text{part.}}\right)\sin\alpha_m x \quad (11)$$

where the integration constants A_m, B_m, C_m, D_m are obtained using boundary conditions written for the sides parallel to x axis, these boundary conditions can be of any type.

For the plate loaded uniformly over the entire surface, using relation (5), and (8) follows:

$$Y_m^{\text{part.}} = \frac{p_m}{\alpha_m^4 D} = \frac{4p}{Dm\pi\alpha_m^4} = \frac{4pa^4}{D\pi^5 m^5}, \quad (12)$$

$m = 1, 3, 5, \dots$.

Further it will be presented the calculation of a square plate (fig. 2) simply supported on the sides $x = 0$ and $x = a$, clamped on the side $y = 0$ and free on the side $y = b$. The dimension of the plate is 6x6 m and it thickness δ = 15 cm. The modulus of elasticity of the material E = 2,31*10^7 kN/m^2 and Poisson ratio ν = 0,2. The plate is loaded with a uniformly distributed load p = 10 kN/m^2.

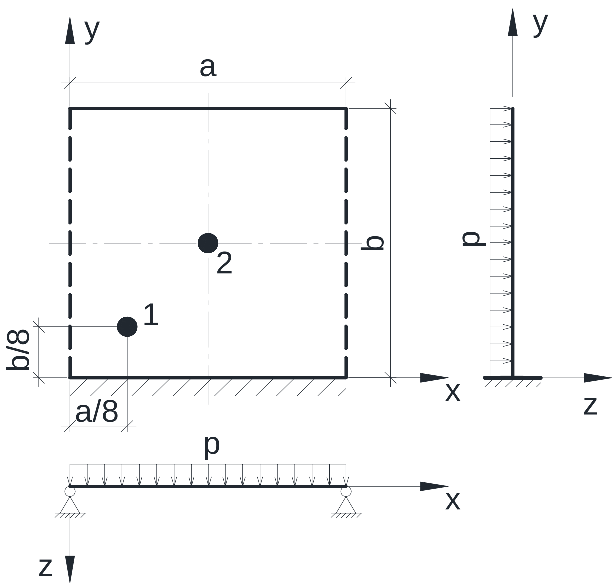

Figure 2. Plate with different boundary conditions.

The boundary conditions are:

- for the clamped side (y = 0) $\begin{cases} w=0 \\ \dfrac{\partial w}{\partial y}=0 \end{cases}$; (13)

- for the simply supported sides (x = 0, x =a)

$$\begin{cases} w=0 \\ M_x=0 \end{cases} \Rightarrow \begin{cases} w=0 \\ \dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2}=0 \end{cases}; \quad (14)$$

- for the free side (y = b)

$$\begin{cases} M_y=0 \\ Q_y^*=0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2}=0 \\ \dfrac{\partial^3 w}{\partial y^3}+(2-\nu)\dfrac{\partial^3 w}{\partial x^2 \partial y}=0 \end{cases}. \quad (15)$$

From the boundary conditions we obtain the integration constants A_m, B_m, C_m, D_m.

$$A_m = -Y_m^{\text{part.}} = -\frac{4pa^4}{D\pi^5 m^5};$$

$$B_m = \frac{(3+\nu)(1-\nu)C^2 + 2\nu C - (1-\nu^2) - \nu(1-\nu)\kappa S}{(3+\nu)(1-\nu)C^2 + (1+\nu)^2 + (1-\nu)^2\kappa^2}\cdot Y_m^{\text{part.}}$$

$$C_m = \frac{(3+\nu)(1-\nu)CS + \nu(1+\nu)S - \nu(1-\nu)\kappa C - (1-\nu)^2\kappa}{(3+\nu)(1-\nu)C^2 + (1+\nu)^2 + (1-\nu)^2\kappa^2}\cdot Y_m^{\text{part.}}$$

$$D_m = -\frac{(3+\nu)(1-\nu)CS + \nu(1+\nu)S - \nu(1-\nu)\kappa C - (1-\nu)^2\kappa}{(3+\nu)(1-\nu)C^2 + (1+\nu)^2 + (1-\nu)^2\kappa^2}\cdot Y_m^{\text{part.}}$$

where $C = \cosh\alpha_m b,\ S = \sinh\alpha_m b,\ \kappa = \alpha_m b$.

Internal efforts expressed through the deflection are obtained with the relations:

$$\left.\begin{aligned} M_x &= D\sum_{m=1}^{\infty}\left(\alpha_m^2 Y_m - \nu Y_m''\right)\sin\alpha_m x; \\ M_y &= D\sum_{m=1}^{\infty}\left(-Y_m'' - \nu\alpha_m^2 Y_m\right)\sin\alpha_m x; \\ M_{xy} &= -D(1-\nu)\sum_{m=1}^{\infty}\alpha_m Y_m' \cos\alpha_m x; \\ Q_x &= D\sum_{n=1}^{\infty}\left(\alpha_m^3 Y_m - \alpha_m Y_m''\right)\cos\alpha_m x; \\ Q_y &= D\sum_{n=1}^{\infty}\left(\alpha_m^2 Y_m' - Y_m'''\right)\sin\alpha_m x; \end{aligned}\right\} \quad (16)$$

The table below presents the results for the investigated plate (fig.2) in the points 1, 2, obtained using Fourier series and the results obtained using finite element method for different mesh and result deviation using FEM from analytical solution.

Mention that the FEM programs (SCAD, ANSYS, Robot, etc.) contain triangular, quadrilateral, etc. elements connected only in nodal points and each node has three degrees of freedom: θ_{xi}, θ_{yi}, w_{xi} .

Table 1. Results obtained using Fourier series and using FEM.

The strain calculated in the given point	**Fourier series**	**Results and deviations obtained using FEM for nxn mesh**			
		4x4		**8x8**	
		Value [kNm/m]	Deviation [%]	Value [kNm/m]	Deviation [%]
M_{x1}	0,35	0,73	109	0,63	80,5
M_{y1}	-6,52	-7,08	8,6	-5,63	-13,7
M_{xy1}	-8,52	-9,38	10,0	-9,0	5,6
M_{x2}	21,0	21,2	0,8	20,0	-4,7
M_{y2}	9,14	10,8	17,9	9,3	1,7
M_{xy2}	0	0	0	0	0

Table 1. (continuation)

The strain calculated in the given point	**Results and deviations obtained using FEM for nxn mesh**			
	16x16		**32x32**	
	Value [kNm/m]	Deviation [%]	Value [kNm/m]	Deviation [%]
M_{x1}	0,45	28,9	0,41	17,5
M_{y1}	-6,1	-6,5	-6,18	5,24
M_{xy1}	-8,77	2,9	-8,71	2,2
M_{x2}	19,7	-6,0	19,7	-6,3
M_{y2}	8,94	-2,2	8,87	-3,0
M_{xy2}	0	0	0	0

The results essentially differ using finite element method when approaching corners of the plates where moments tend to infinity. Similar results are obtained for plates of different sizes and loads.

There is no warning that existing elements in different computer programs (SCAD, Lira, ANSYS etc.) doesn't allow to describe conditions more complicated than clamped support. For free sides, in general, boundary conditions can't be described using elements with three degrees of freedom in the node.

CONCLUSIONS

1. For stress calculations, the existing computing programs (SCAD, Lira, ANSYS etc.) are not able to give accurate results to satisfy all cases of boundary conditions except the clamped side.

2. Plate corner points are singular points for which the finite element method should use special elements that would take into account the behavior of the solution in this points.

3. To perform calculations with finite element method with high accuracy there were developed special finite elements with an increased number of degrees of freedom in nodes, but implementing them in computer programs is difficult, so they are missing.

4. An alternative to FEM is boundary element method (BEM) which is free from the mentioned gaps, because the implementation of special elements in the method is more simple. The contour elements that will be implemented will allow us to satisfy different boundary conditions. These elements can be used including the asymptotic behavior of solutions in singular points.

The author expresses sincere thanks to his supervisor Dr. hab. prof. Gheorghe Moraru who contributed with his observations to improve this paper.

Bibliography

1. ***Konchakowskij Z.*** *Plity'. Staticheskie raschety', Moskva, Strojizdat, 1984.*

2. ***Moraru Gh.***, *Introducere în metoda elementelor finite şi de frontieră*, Chişinău, U.T.M., 2002.

3. ***Perel'muter A., Slivker V.*** *Raschetny'e modeli sooruzhenij i vozmozhnost' ih analiza, Kiev, Stal', 2002.*

4. ***Timoshenko S., şi Woinowsky-Krieger S.*** *Theory of plates and shells, New York, McGraw-Hill,1959.*

Recommended for publication: 07.03.2013.