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AN ALGORITHM FOR DETECTION OF ULTRA-WEAK COMPLETENESS OF THE SYSTEMS OF FORMULAS IN A PROVABILITY LOGIC OF PROPOSITIONS

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Abstract We consider the problem of completeness relative to ultra-weak expressibility of the systems of formulas in the simplest non-trivial extension of the propositional provability logic. We propose an algorithm to address this problem. It is a first step toward investigation of this problem in propositional provability logic. A similar problem was considered by prof. Mefodie Raţă in the case of the intuitionistic logic.

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1. INTRODUCTION

The criteria of completeness with respect to expressibility is well-known in the case of boolean functions [1, 2]. A. V. Kuznetsov [3, 4] has specified the notion of expressibility to the case of formulas in logical calculi, using the rule of replacement by its equivalent in the given logic. Professor Mefodie Raţă has obtained the criterion of completeness relative to expressibility in propositional intuitionistic logic and its extensions [5, 6] and in [7, p.15] he also considered the notion of ultra-weak expressibility.

We consider the simplest non-classical 4-valued extension of the propositional provability logic of Gödel-Löb GL [8] and found out the necessary and sufficient conditions for a system of formulas to be complete relative to ultra-weak expressibility of formulas of this logic. An algorithm based on this finding is proposed.

2. DEFINITIONS AND NOTATIONS

Propositional provability logic GL. The formulas of the propositional provability calculus of GL are built from the symbols of propositional variables $p, q, r, \dots$ (may be also indexed), by means of the symbols of logical connectives $\&, \vee, \supset, \neg$ and Δ (represent the unary modal operation of provability by Gödel), and parentheses. For example, the expressions $(p\&\neg p)$, $(p \supset p)$, $(\Delta(p\&\neg p))$ and $(\neg(\Delta(p\&\neg p)))$ are formulas in the calculus of GL, representing the constant formulas denoted in the following by $0, 1, \sigma, \rho$, and we denote the formulas $(p\&\Delta p)$ and $((p \supset q)\&(q \supset p))$ as $\Box p$ (box p) and $(p \sim q)$ (equivalence of p and q). External parentheses are usually omitted. The calculus of the GL is determined by the axioms of the classical calculus of propositions, three Δ-axioms

$$\Delta(p \supset q) \supset (\Delta p \supset \Delta q),\ \Delta(\Delta p \supset p) \supset \Delta p,\ \Delta p \supset \Delta\Delta p$$

and the next three rules of inference: 1) the rule of substitution, 2) the modus ponens rule, and 3) the rule of necessitation which allows to pass from formula A to formula ΔA. The notions of theorems and the logic of the given calculus are defined as usual [8].

An extension L_2 of the logic L_1 is any set of formulas of the calculus of L_1 containing all axioms of L, is closed relative to the rules of inference of L and $L_1 \subseteq L_2$ (as sets).

Magari's algebras. A Magari's algebra [9] (also referred to as diagonalizable algebra) $\mathfrak{D}$ is a boolean algebra $\mathfrak{B} = (B; \&, \vee, \supset, \neg, \mathbf{0}, \mathbf{1})$ with an additional operator Δ satisfying the following identities:

$$\begin{aligned} \Delta\mathbf{1} &= \mathbf{1}, \\ \Delta(x\&y) &= (\Delta x\&\Delta y), \\ \Delta(\Delta x \supset x) &\leq \Delta x \end{aligned}$$

where $\mathbf{1}$ is the unit of $\mathfrak{B}$.

Interpreting logical connectives of a formula F by corresponding operations on a Magari's algebra $\mathfrak{D}$ we can evaluate any formula of GL on any algebra $\mathfrak{D}$. If for any evaluation of variables of F by elements of $\mathfrak{D}$ the resulting value of the formula F on $\mathfrak{D}$ is $\mathbf{1}$ they say F *is valid on* $\mathfrak{D}$. The set of all valid formulas on the given Magari's algebra $\mathfrak{D}$ is an extension of GL [10], also called the logic of the algebra $\mathfrak{D}$, and denoted by $L\mathfrak{D}$.

We consider the 4-valued Magari's algebra $\mathfrak{B}_2 = (\{\mathbf{0}, \rho, \sigma, \mathbf{1}\}; \&, \vee, \supset, \neg, \Delta)$, its boolean operations $\&, \vee, \supset, \neg$ are defined as usual, and the operation Δ is defined as:

$$\Delta\mathbf{0} = \Delta\rho = \sigma,\ \Delta\sigma = \Delta\mathbf{1} = \mathbf{1}.$$

In the following we consider the logic $L\mathfrak{B}_2$.

Ultra-weak expressibility of formulas. Suppose in the logic L we can define the equivalence of two formulas. The formula F is said to be a constant in the logic L if for any variables ϖ and σ the formula $F[\varpi/\sigma]$ is equivalent to F. The formula F is said to be (explicitly) expressible via a system of formulas Σ in the logic L if F can be obtained from variables and formulas of Σ using two rules: a) the rule of weak substitution, which allows to pass from two formulas, say A and B, to the result of substitution of one of them in another in place of all occurrences of any variable p of the formula $\frac{A,B}{A[p/B]}$ (where we denote by $A[p/B]$ the thought substitution); b) the rule of passing to an equivalent formula in L which states that if we have already get formula A and we know A is equivalent in L to B, then we have also formula B [11].

The formula F is said to be ultra-weakly expressible in L via Σ if it can be obtained from unary formulas and Σ via already mentioned above rules of weak substitution and passing to equivalent.

The system Σ is said to be ultra-weak complete in the logic L if any formula of the calculus of L is ultra-weak expressible in L via Σ [6, p. 15].

Relations on algebras. They say [6] the formula $F(p_1, \dots, p_n)$ preserves on the Magari's algebra $\mathfrak{D}$ the relation $R(x_1, \dots, x_m)$ if for any elements $\alpha_{11}, \dots, \alpha_{mn}$ of $\mathfrak{D}$ the relations

$$R(\alpha_{1j}, \dots, \alpha_{mj}), \quad j = 1, \dots, n$$

implies

$$R(F(\alpha_{11}, \dots, \alpha_{1n}), \dots, F(\alpha_{m1}, \dots, \alpha_{mn}))$$

The relation $R(x_1, \dots, x_m)$ on a finite algebra $\mathfrak{D}$ can be substituted by a corresponding matrix β_{ik} $(i = 1, \dots, m,\ k = 1, \dots, l)$ of all elements of $\mathfrak{D}$ such that the statement $R(\beta_{1k}, \dots, \beta_{mk})$ holds [7]. In this case we speak about preserving of a matrix instead of preserving of a relation on $\mathfrak{D}$.

3. PRELIMINARY RESULTS

Representation of operations on $\{\mathbf{0}, \rho, \sigma, \mathbf{1}\}$ by formulas. Let us recall some results mentioned in [12, 13].

Theorem 3.1. *A function $f : \{\mathbf{0}, \rho, \sigma, \mathbf{1}\}^n \to \{\mathbf{0}, \rho, \sigma, \mathbf{1}\}$ $(n = 0, 1, \dots)$ can be represented by a formula of the calculus of the logic $L\mathfrak{B}_2$ if and only if it conserves the relation $\Delta x = \Delta y$ on the algebra $\mathfrak{B}_2$.*

Next statement is a consequence of the above theorem [13, Proposition 3.1].

Propoziia 3.1. *There are 64 unary formulas in the calculus of the logic $L\mathfrak{B}_2$ which are not equivalent each other in $L\mathfrak{B}_2$ and realize the corresponding unary operations of the algebra $\mathfrak{B}_2$.*

Table 1 Unary operations of $\mathfrak{B}_2$

p	I_{1j}	I_{2j}	I_{3j}	I_{4j}	I_{5j}	I_{6j}	I_{7j}	I_{8j}
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	ρ	ρ	σ	σ	$\mathbf{1}$	$\mathbf{1}$
ρ	$\mathbf{0}$	ρ	$\mathbf{0}$	ρ	σ	$\mathbf{1}$	σ	$\mathbf{1}$
p	I_{i1}	I_{i2}	I_{i3}	I_{i4}	I_{i5}	I_{i6}	I_{i7}	I_{i8}
σ	$\mathbf{0}$	$\mathbf{0}$	ρ	ρ	σ	σ	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	ρ	$\mathbf{0}$	ρ	σ	$\mathbf{1}$	σ	$\mathbf{1}$

In order to describe the derived unary operations of the algebra $\mathfrak{B}_2$ we use the table 1, where $I_{ij}(p)$ $(i = 1, \ldots, 8;\ j = 1, \ldots, 8)$ denotes the unary operation which for $p = \mathbf{0}$ and $p = \rho$ takes values from the i-th column, and for $p = \sigma$ and $p = \mathbf{1}$ it takes values from the j-th column.

For example, $I_{11} = \mathbf{0}$, $I_{16} = p$, $I_{73} = \neg p$, $I_{58} = \Delta p$, $I_{88} = \mathbf{1}$.

4. MAIN RESULT

Consider the following relations on $\mathfrak{B}_2$ (read symbols "==" as "defined by"):

1) $R_1(x, y, z, u) == ((\Delta x = \Delta y)\&(\Delta z = \Delta u)\&(\Delta x = \Delta z)\ \&\ ((x \sim y) = (z \sim u)))$;

2) $R_2(x, y, z, u) == ((\Delta x = \Delta y)\&(\Delta z = \Delta u)\&((x = y) \vee (z = u) \vee (\Delta x = \Delta z)))$;

3) $R_3(x, y, z, u) == (\Delta(x \sim y) = \Delta(z \sim u))$.

We denote by $\mathfrak{M}_i$ the corresponding matrix to the relation R_i on the algebra $\mathfrak{B}_2$ and denote with Π_i the class of all formulas, which preserves the relation R_i on the algebra $\mathfrak{B}_2$, i.e. the class of all formulas, which conserves the matrix $\mathfrak{M}_i$ on $\mathfrak{B}_2$ for any $i = 1, 2, 3$.

The table 2 presents the list of all classes Π_1, Π_2, Π_3 and their corresponding matrix.

Table 2: The class of formulas and the corresponding matrix

The class	Defining matrix
Π_1	$\begin{pmatrix} \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{000}\rho\rho\rho\ \ \sigma\sigma\sigma\mathbf{111} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{0}\rho\rho\mathbf{00}\rho\ \ \sigma\mathbf{11}\sigma\sigma\mathbf{1} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\ \ \mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \rho\rho\mathbf{0}\rho\mathbf{00}\ \ \mathbf{11}\sigma\mathbf{1}\sigma\sigma \end{pmatrix}$

The table 2 continues on the next page

The next part of the table 2

The class	Defining matrix
Π_2	$\begin{pmatrix} \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{0000000}\rho\rho\rho\rho\rho\rho\rho\sigma\sigma\sigma\sigma\sigma\sigma\sigma\mathbf{1111111} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\sigma\sigma\sigma\mathbf{1111}\sigma\sigma\sigma\sigma\mathbf{111} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{0}\rho\rho\mathbf{00}\rho\rho\mathbf{00}\rho\rho\mathbf{00}\rho\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{1} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{01}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{0000}\rho\rho\rho\rho\sigma\sigma\sigma\sigma\mathbf{1111}\sigma\mathbf{1}\sigma\mathbf{10}\rho\mathbf{0}\rho \\ \mathbf{0000}\rho\rho\rho\rho\sigma\sigma\sigma\sigma\mathbf{11111}\sigma\mathbf{1}\sigma\rho\mathbf{0}\rho\mathbf{0} \\ \sigma\sigma\mathbf{11}\sigma\sigma\mathbf{1100}\rho\rho\mathbf{00}\rho\rho\mathbf{00}\rho\rho\sigma\sigma\mathbf{11} \\ \sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{10}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{00}\rho\rho\sigma\sigma\mathbf{11} \end{pmatrix}$
Π_3	$\begin{pmatrix} \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{0000000}\rho\rho\rho\rho\rho\rho\rho\sigma\sigma\sigma\sigma\sigma\sigma\sigma\mathbf{1111111} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\sigma\sigma\sigma\mathbf{1111}\sigma\sigma\sigma\sigma\mathbf{111} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \mathbf{0}\rho\rho\mathbf{00}\rho\rho\mathbf{00}\rho\rho\mathbf{00}\rho\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{1} \\ \mathbf{0}\rho\sigma\mathbf{1}\ \ \rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{0}\rho\mathbf{01}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{00000000}\rho\rho\rho\rho\rho\rho\rho\rho\mathbf{00000000}\rho\rho\rho\rho\rho\rho\rho\rho \\ \mathbf{0000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11} \\ \sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{110000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho \\ \sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathbf{00000000}\rho\rho\rho\rho\rho\rho\rho\rho\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11} \\ \sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{1100000000}\rho\rho\rho\rho\rho\rho\rho\rho \\ \sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{10000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho \\ \mathbf{0000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11} \\ \mathbf{00000000}\rho\rho\rho\rho\rho\rho\rho\rho\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1}\sigma\mathbf{1} \\ \sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{11}\sigma\sigma\mathbf{1100000000}\rho\rho\rho\rho\rho\rho\rho\rho \\ \mathbf{0000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho\mathbf{0000}\rho\rho\rho\rho \end{pmatrix}$

Consider now the functions on support of $\mathfrak{B}_2$ defined in the table 3.

Let us note that everyone of these functions conserve the relation $\Delta x = \Delta y$ on the algebra $\mathfrak{B}_2$ and according to theorem 3.1 there are formulas D_1, D_2, D_3, D_4 which represent them on the algebra $\mathfrak{B}_2$.

Now we are ready to formulate the main result of the paper.

Theorem 4.1. *The system of formulas Σ is ultra-weak complete in the logic $L\mathfrak{B}_2$ if and only if for every class of formulas Π_1, Π_2, Π_3 there exists a corresponding formula $F_i, i = 1, 2, 3,$ in Σ such that $F_i \notin \Pi_i$.*

Proof. Every one can verify that the classes of formulas Π_1, Π_2, Π_3 are closed relative to ultra-weak expressibility in the logic $L\mathfrak{B}_2$ and each one of them

Table 3 The tables of the functions f_1, f_2, f_3, f_4

p	**0**	**0**	**0**	**0**	ρ	ρ	ρ	ρ	σ	σ	σ	σ	**1**	**1**	**1**	**1**
q	**0**	ρ	σ	**1**	**0**	ρ	σ	**1**	**0**	ρ	σ	**1**	**0**	ρ	σ	**1**
f_1	**1**	σ	ρ	**0**	σ	**1**	**0**	ρ	ρ	**0**	**1**	σ	**0**	ρ	σ	**1**
f_2	ρ	ρ	ρ	ρ	**0**	**0**	**0**	**0**	**1**	**1**	σ	**1**	σ	σ	σ	σ
f_3	**0**	ρ	**1**	**1**	ρ	ρ	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
f_4	**0**	ρ	σ	**1**	**0**	ρ	σ	**1**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

does not contain the formula $((p\&q) \vee (p\&r) \vee (q\&r))$. Also we can verify that the following relations hold

$$\{D_1, D_4\} \subseteq \Pi_1, \qquad \{D_2, D_3\} \cap \Pi_1 = \emptyset,$$
$$\{D_3, D_4\} \subseteq \Pi_2, \qquad \{D_1, D_2\} \cap \Pi_2 = \emptyset,$$
$$\{D_1, D_2\} \subseteq \Pi_3, \qquad \{D_3, D_4\} \cap \Pi_3 = \emptyset.$$

The necessary part of the theorem follows from the fact that the classes of formulas Π_1, Π_2, Π_3 are closed relative to ultra-weak expressibility in the logic $L\mathfrak{B}_2$ and are pairwise distinct according to above relations. So they are not ultra-weakly complete in the logic $L\mathfrak{B}_2$.

Now consider the sufficient part of the theorem. Suppose the system of formulas Σ is ultra-weakly complete in the logic $L\mathfrak{B}_2$. Suppose Σ contains a system of formulas $\{F_1, F_2, F_3\}$ which do not belong to the corresponding classes Π_1, Π_2, Π_3 and do not contain other variables excepting $p_1, \ldots, p_n$. It is not supposed the formulas F_1, F_2, F_3 are distinct. It is sufficiently now to prove that conjunction $(p\&q)$ is ultra-weakly expressible in $L\mathfrak{B}_2$ via $\{F_1, F_2, F_3\}$. The continuation of the proof is presented in the next lemmas 4.1, 4.2, 4.3 and 4.4. ∎

Lemma 4.1. *Formulas $C(p, q)$ and $D(p, q)$ satisfying conditions*

$$\begin{gathered} C[\mathbf{0},\mathbf{0}] = C[\mathbf{0},\rho] = C[\rho,\mathbf{0}] = \mathbf{0},\ C[\rho,\rho] = \rho \\ D[\sigma,\sigma] = C[\sigma,\mathbf{1}] = D[\mathbf{1},\sigma] = \sigma,\ D[\mathbf{1},\mathbf{1}] = \mathbf{1}. \end{gathered} \tag{1}$$

are ultra-weak expressible in $L\mathfrak{B}_2$ via formula F_1.

Proof. Consider formula F_1, which do not conserve the relation R_1 on $\mathfrak{B}_2$. Then there exist four ordered sets of elements $< \alpha_1, \ldots, \alpha_n >$, $< \beta_1, \ldots, \beta_n >$,

$< \gamma_1, \ldots, \gamma_n >$ and $< \delta_1, \ldots, \delta_n >$ from $\mathfrak{B}_2$ such that

$$R_1(\alpha_i, \beta_i, \gamma_i, \delta_i),\ i = 1, \ldots, n \tag{2}$$

$$\begin{pmatrix} F_1[\alpha_1, \ldots, \alpha_n] \\ F_1[\beta_1, \ldots, \beta_n] \\ F_1[\gamma_1, \ldots, \gamma_n] \\ F_1[\delta_1, \ldots, \delta_n] \end{pmatrix} \subseteq M, \tag{3}$$

where

$$M = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \rho & \rho & \rho & \rho & \sigma & \sigma & \sigma & \sigma & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \rho & \rho & \mathbf{0} & \mathbf{0} & \rho & \rho & \sigma & \sigma & \mathbf{1} & \mathbf{1} & \sigma & \sigma & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \rho & \mathbf{0} & \rho & \mathbf{0} & \rho & \mathbf{0} & \rho & \sigma & \mathbf{1} & \sigma & \mathbf{1} & \sigma & \mathbf{1} & \sigma & \mathbf{1} \\ \rho & \mathbf{0} & \mathbf{0} & \rho & \mathbf{0} & \rho & \rho & \mathbf{0} & \mathbf{1} & \sigma & \sigma & \mathbf{1} & \sigma & \mathbf{1} & \mathbf{1} & \sigma \end{pmatrix}.$$

The right-hand side of the relation (3) determines 16 possible cases for F_1. Consider formula $B(p_1, \ldots, p_n)$, defined by the scheme

$$B(p_1, \ldots, p_n) = \begin{cases} I_{23}[F_1], & \text{if } F_1[\alpha_1, \ldots, \alpha_n] \in \{\mathbf{0}, \mathbf{1}\}, \\ I_{32}[F_1], & \text{if } F_1[\alpha_1, \ldots, \alpha_n] \in \{\rho, \sigma\} \end{cases}$$

Formula B is ultra-weak expressible via formula F_1. It is easy to verify that

$$\begin{pmatrix} B[\alpha_1, \ldots, \alpha_n] \\ B[\beta_1, \ldots, \beta_n] \\ B[\gamma_1, \ldots, \gamma_n] \\ B[\delta_1, \ldots, \delta_n] \end{pmatrix} \subseteq \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \rho & \rho \\ \mathbf{0} & \rho & \mathbf{0} & \rho \\ \rho & \mathbf{0} & \mathbf{0} & \rho \end{pmatrix}. \tag{4}$$

Denote elements $B[\alpha_1, \ldots, \alpha_n]$, $B[\beta_1, \ldots, \beta_n]$, $B[\gamma_1, \ldots, \gamma_n]$ and $B[\delta_1, \ldots, \delta_n]$ by correspondent letters α, β, γ and δ. Build formula $B'(p, q, r) = B[B'_1, \ldots, B'_n]$, where for any $i = 1, \ldots, n$

$$B'_i(p, q, r) = \mathbf{0},\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$

$$p,\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \rho,\ \delta_i = \rho,$$

$$q,\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$

$$r,\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$

$$I_{37}[r],\ \text{if } \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$

$$I_{37}[q],\ \text{if } \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$

$$I_{37}[p],\ \text{if } \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$

$$\rho,\ \text{if } \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \rho,\ \delta_i = \rho,$$

$$\sigma,\ \text{if } \alpha_i=\sigma,\ \beta_i=\sigma,\ \gamma_i=\sigma,\ \delta_i=\sigma,$$

$$I_{63}(p),\ \text{if } \alpha_i=\sigma,\ \beta_i=\sigma,\ \gamma_i=\mathbf{1},\ \delta_i=\mathbf{1},$$

$$I_{63}[q],\ \text{if } \alpha_i=\sigma,\ \beta_i=\mathbf{1},\ \gamma_i=\sigma,\ \delta_i=\mathbf{1},$$

$$I_{63}[r],\ \text{if } \alpha_i=\sigma,\ \beta_i=\mathbf{1},\ \gamma_i=\mathbf{1},\ \delta_i=\sigma,$$

$$I_{72}[r],\ \text{if } \alpha_i=\mathbf{1},\ \beta_i=\sigma,\ \gamma_i=\sigma,\ \delta_i=\mathbf{1},$$

$$I_{72}[q],\ \text{if } \alpha_i=\mathbf{1},\ \beta_i=\sigma,\ \gamma_i=\mathbf{1},\ \delta_i=\sigma,$$

$$I_{72}(p),\ \text{if } \alpha_i=\mathbf{1},\ \beta_i=\mathbf{1},\ \gamma_i=\sigma,\ \delta_i=\sigma,$$

$$\mathbf{1},\ \text{if } \alpha_i=\mathbf{1},\ \beta_i=\mathbf{1},\ \gamma_i=\mathbf{1},\ \delta_i=\mathbf{1},$$

(there are no other cases for α_i, β_i, γ_i, δ_i). $B'(p,q,r)$ is ultra-weak expressible via formula B. Obviously, by relations (2) and (3), we get $B'_i[\mathbf{0},\mathbf{0},\mathbf{0}]=\alpha_i$, $B'_i[\mathbf{0},\rho,\rho]=\beta_i$, $B'_i[\rho,\mathbf{0},\rho]=\gamma_i$ şi $B'_i[\rho,\rho,\mathbf{0}]=\delta_i$. Taking in account (4), we obtain

$$\begin{pmatrix} B'[\mathbf{0},\mathbf{0},\mathbf{0}] \\ B'[\mathbf{0},\rho,\rho] \\ B'[\rho,\mathbf{0},\rho] \\ B'[\rho,\rho,\mathbf{0}] \end{pmatrix} \subseteq \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \rho & \rho \\ \mathbf{0} & \rho & \mathbf{0} & \rho \\ \rho & \mathbf{0} & \mathbf{0} & \rho \end{pmatrix}. \tag{5}$$

Examine formula $A(p,q,r)$ defined as

$$A(p,q,r)=\begin{cases} B'(p,q,r), & \text{if } \alpha=\beta=\gamma=\mathbf{0},\ \delta=\rho, \\ B'[p,I_{37}[q],I_{37}[r]], & \text{if } \alpha=\beta=\delta=\mathbf{0},\ \gamma=\rho, \\ B'[I_{37}(p),q,I_{37}[r]], & \text{if } \alpha=\gamma=\delta=\mathbf{0},\ \beta=\rho, \\ I_{37}[B'[I_{37}(p),I_{37}[q],r]], & \text{if } \beta=\gamma=\delta=\rho,\ \alpha=\mathbf{0}. \end{cases}$$

Formula $A(p,q,r)$ is ultra-weak expressible via formulas of lemma. Then, by relations (5), we obtain

$$A[\mathbf{0},\mathbf{0},\mathbf{0}]=A[\mathbf{0},\rho,\rho]=A[\rho,\mathbf{0},\rho]=\mathbf{0},\ A[\rho,\rho,\mathbf{0}]=\rho. \tag{6}$$

Since A conserves relation $\Delta x=\Delta y$ on $\mathfrak{B}_2$, it follows that

$$A[\mathbf{0},\rho,\mathbf{0}]\in\{\mathbf{0},\rho\},\ A[\rho,\mathbf{0},\mathbf{0}]\in\{\mathbf{0},\rho\}.$$

Thus, taking into account equalities (6), there are 4 possible sub-cases for formula A. In each of these sub-cases formula $C(p,q)$ is ultra-weak expressible via formulas from lemma and $A(p,q,r)$ in the following way:

1) If $A[\mathbf{0},\rho,\mathbf{0}]=\mathbf{0}$, $A[\rho,\mathbf{0},\mathbf{0}]=\mathbf{0}$, then $C(p,q)=A[p,q,\mathbf{0}]$.
2) If $A[\mathbf{0},\rho,\mathbf{0}]=\mathbf{0}$, $A[\rho,\mathbf{0},\mathbf{0}]=\rho$, then $C(p,q)=A[A_5(p),p,q]$.
3) If $A[\mathbf{0},\rho,\mathbf{0}]=\rho$, $A[\rho,\mathbf{0},\mathbf{0}]=\mathbf{0}$, then $C(p,q)=A[p,A_5(p),q]$.
4) If $A[\mathbf{0},\rho,\mathbf{0}]=\rho$, $A[\rho,\mathbf{0},\mathbf{0}]=\rho$, then $C(p,q)=A[A_5(p),q,A_5[q]]$.

To finalize the proof of the lemma let us note that $D(p,q) = I_{62}[C[I_{62}(p), I_{62}[q]]]$. Lemma 4.1 is proved. ∎

Lemma 4.2. *A formula $S(p,q)$ satisfying conditions*

$$S[\mathbf{0},\mathbf{0}] = \mathbf{0},\ S[\mathbf{0},\rho] = \rho,\ S[\sigma,\mathbf{1}] = \sigma,\ S[\mathbf{1},\mathbf{1}] = \mathbf{1}, \tag{7}$$

is ultra-weak expressible in the logic $L\mathfrak{B}_2$ via any formula F_2.

Proof. Consider formula F_2. Since it does not conserve the relation R_2 on $\mathfrak{B}_2$ there are ordered sets of elements $< \alpha_1, \dots, \alpha_n >$, $< \beta_1, \dots, \beta_n >$, $< \gamma_1, \dots, \gamma_n >$ and $< \delta_1, \dots, \delta_n >$ from $\mathfrak{B}_2$ such that

$$R_2(\alpha_i, \beta_i, \gamma_i, \delta_i),\ i = i, \dots, n \tag{8}$$

$$\begin{pmatrix} F_2[\alpha_1,\dots,\alpha_n] \\ F_2[\beta_1,\dots,\beta_n] \\ F_2[\gamma_1,\dots,\gamma_n] \\ F_2[\delta_1,\dots,\delta_n] \end{pmatrix} \subseteq \begin{pmatrix} \mathbf{0} & \mathbf{0} & \rho & \rho & \sigma & \sigma & \mathbf{1} & \mathbf{1} \\ \rho & \rho & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \sigma & \sigma \\ \sigma & \mathbf{1} & \sigma & \mathbf{1} & \mathbf{0} & \rho & \mathbf{0} & \rho \\ \mathbf{1} & \sigma & \mathbf{1} & \sigma & \rho & \mathbf{0} & \rho & \mathbf{0} \end{pmatrix}. \tag{9}$$

The right-hand side of the inclusion (9) define 8 cases to take into account. Observe that the matrix $\mathfrak{M}_2$ corresponding to the relation R_2 on $\mathfrak{B}_2$ is invariant relative to any permutation of rows. So we can consider that the last seven cases can be reduced to the first one when

$$\left.\begin{aligned} F_2[\alpha_1,\dots,\alpha_n] &= \mathbf{0}, \\ F_2[\beta_1,\dots,\beta_n] &= \rho, \\ F_2[\gamma_1,\dots,\gamma_n] &= \sigma, \\ F_2[\delta_1,\dots,\delta_n] &= \mathbf{1}. \end{aligned}\right\} \tag{10}$$

Build formula

$$E(p_1,\dots,p_{36}) = F_2[E_1(p_1,\dots,p_{36}),\dots,E_n(p_1,\dots,p_{36})],$$

where for any $i = 1, \dots, n$

$$E_i = \mathbf{0},\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$

$$E_i = C[p_1,p_2],\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$

$$E_i = C[p_1,p_3],\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$

$$E_i = p_1,\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \rho,\ \delta_i = \rho,$$

$$E_i = p_{13},\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \sigma,\ \delta_i = \sigma,$$

$$E_i = p_{14},\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \sigma,\ \delta_i = \mathbf{1},$$

$$E_i = p_{15},\ \text{if } \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{1},\ \delta_i = \sigma,$$

$$E_i = p_{16},\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = C[p_2, p_6],\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$
$$E_i = p_2,\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$
$$E_i = p_3,\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$
$$E_i = I_{37}[C[p_4, p_5]],\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = p_{17},\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = p_{18},\ \text{if}\ \alpha_i = \mathbf{0},\ \beta_i = \rho,\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = C[p_4, p_5],\ \text{if}\ \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$
$$E_i = p_4,\ \text{if}\ \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$
$$E_i = p_5,\ \text{if}\ \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$
$$E_i = I_{37}[C[p_2, p_6]],\ \text{if}\ \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = p_{19},\ \text{if}\ \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = p_{20},\ \text{if}\ \alpha_i = \rho,\ \beta_i = \mathbf{0},\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = p_6,\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$
$$E_i = I_{37}[C[p_1, p_3]],\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$
$$E_i = I_{37}[C[p_1, p_3]], \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$
$$E_i = \rho,\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = p_{21},\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = p_{22},\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \sigma,\ \delta_i = \mathbf{1},$$
$$E_i = p_{23},\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \mathbf{1},\ \delta_i = \sigma,$$
$$E_i = p_{24},\ \text{if}\ \alpha_i = \rho,\ \beta_i = \rho,\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = p_{25},\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$
$$E_i = p_{26},\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$
$$E_i = p_{27},\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$
$$E_i = p_{28},\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = \sigma,\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = I_{63}[C[p_1, p_2]],\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \sigma,\ \delta_i = \mathbf{1},$$
$$E_i = I_{63}[C[p_1, p_3]],\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \mathbf{1},\ \delta_i = \sigma,$$
$$E_i = p_7,\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \sigma,\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = p_{29},\ \text{if}\ \alpha_i = \sigma,\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$

$$E_i = p_{30},\ \text{if } \alpha_i = \sigma,\ \beta_i = \mathbf{1},\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = I_{63}[C[p_2, p_6]],\ \text{if } \alpha_i = \sigma,\ \beta_i = \mathbf{1},\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = p_8,\ \text{if } \alpha_i = \sigma,\ \beta_i = \mathbf{1},\ \gamma_i = \sigma,\ \delta_i = \mathbf{1},$$
$$E_i = p_9,\ \text{if } \alpha_i = \sigma,\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{1},\ \delta_i = \sigma,$$
$$E_i = I_{73}[C[p_4, p_5]],\ \text{if } \alpha_i = \sigma,\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = p_{31},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \sigma,\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$
$$E_i = p_{32},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \sigma,\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = I_{63}[C[p_4, p_5]],\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \sigma,\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = p_{10},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \sigma,\ \gamma_i = \sigma,\ \delta_i = \mathbf{1},$$
$$E_i = p_{11},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \sigma,\ \gamma_i = \mathbf{1},\ \delta_i = \sigma,$$
$$E_i = I_{73}[C[p_2, p_6]],\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \sigma,\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$
$$E_i = p_{33},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{0},\ \delta_i = \mathbf{0},$$
$$E_i = p_{34},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{0},\ \delta_i = \rho,$$
$$E_i = p_{35},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \rho,\ \delta_i = \mathbf{0},$$
$$E_i = p_{36},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \rho,\ \delta_i = \rho,$$
$$E_i = p_{12},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \sigma,\ \delta_i = \sigma,$$
$$E_i = I_{73}[C[p_1, p_3]],\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \sigma,\ \delta_i = \mathbf{1},$$
$$E_i = I_{73}[C[p_1, p_2]],\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{1},\ \delta_i = \sigma,$$
$$E_i = \mathbf{1},\ \text{if } \alpha_i = \mathbf{1},\ \beta_i = \mathbf{1},\ \gamma_i = \mathbf{1},\ \delta_i = \mathbf{1},$$

(there are no other cases for α_i, β_i, γ_i and δ_i according to relation (8)). Formula E is ultra-weak expressible via formulas $C(p, q)$ and F_2. Note that

$$\begin{aligned}
&E_i[\mathbf{0},\mathbf{0},\mathbf{0},\rho,\rho,\rho,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\\
&\quad \rho,\rho,\rho,\rho,\rho,\rho,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}] = \alpha_i,\\
&E_i[\mathbf{0},\rho,\rho,\mathbf{0},\mathbf{0},\rho,\sigma,\mathbf{1},\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\rho,\rho,\\
&\quad \mathbf{0},\mathbf{0},\rho,\rho,\rho,\rho,\sigma,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}] = \beta_i,\\
&E_i[\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\sigma,\mathbf{1},\\
&\quad \sigma,\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\rho,\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{0},\rho,\rho] = \gamma_i,\\
&E_i[\rho,\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\sigma,\mathbf{1},\sigma,\sigma,\sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\\
&\quad \sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,] = \delta_i.
\end{aligned} \tag{11}$$

Taking into account (10), last equalities (11) and the design of E we can conclude that E satisfies conditions (12) below.

$$\left.\begin{array}{l}
E[\mathbf{0},\mathbf{0},\mathbf{0},\rho,\rho,\rho,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\\
\quad \rho,\rho,\rho,\rho,\rho,\rho,\sigma,\sigma,\sigma,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}]=\mathbf{0},\\
E[\mathbf{0},\rho,\rho,\mathbf{0},\mathbf{0},\rho,\sigma,\mathbf{1},\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\rho,\rho,\\
\quad \mathbf{0},\mathbf{0},\rho,\rho,\rho,\rho,\sigma,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}]=\rho,\\
E[\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\sigma,\mathbf{1},\\
\quad \sigma,\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\rho,\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{0},\rho,\rho]=\sigma,\\
E[\rho,\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\sigma,\mathbf{1},\sigma,\sigma,\sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\\
\quad \sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,]=\mathbf{1},
\end{array}\right\} \qquad (12)$$

It is not so difficult to verify that formula

$$\begin{aligned}Z(p_1,\ldots,p_{24}) = E[&I_{14},p_1,p_2,p_3,p_4,I_{41},I_{58},p_5,p_6,p_7,p_8,I_{85},\\
&I_{15},p_9,p_{10},I_{18},p_{11},p_{12},p_{13},p_{14},I_{45},p_{15},p_{16},\\
&I_{48},I_{51},p_{17},p_{18},I_{54},p_{19},p_{20},p_{21},p_{22},I_{81},p_{23},\\
&p_{24},I_{84}].\end{aligned}$$

satisfies restrictions:

$$\begin{aligned}
&Z[\mathbf{0},\mathbf{0},\rho,\rho,\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\\
&\qquad\qquad \rho,\rho,\rho,\rho,\sigma,\sigma,\sigma,\sigma,\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}]=\mathbf{0},\\
&Z[\rho,\rho,\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1},\sigma,\sigma,\mathbf{0},\mathbf{0},\rho,\rho,\\
&\qquad\qquad \mathbf{0},\mathbf{0},\rho,\rho,\sigma,\sigma,\mathbf{1},\mathbf{1},\sigma,\sigma,\mathbf{1},\mathbf{1}]=\rho,\\
&Z[\mathbf{0},\rho,\mathbf{0},\rho,\sigma,\mathbf{1},\sigma,\mathbf{1},\mathbf{1},\sigma,\mathbf{1},\sigma,\\
&\qquad\qquad \mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho,\mathbf{0},\rho]=\sigma,\\
&Z[\rho,\mathbf{0},\rho,\mathbf{0},\mathbf{1},\sigma,\mathbf{1},\sigma,\mathbf{1},\sigma,\sigma,\mathbf{1},\\
&\qquad\qquad \sigma,\mathbf{1},\mathbf{1},\sigma,\rho,\rho,\mathbf{0},\mathbf{0},\rho,\mathbf{0},\rho,\rho,\mathbf{0}]=\mathbf{1}.
\end{aligned}$$

Now it is really easy to verify that

$$\begin{aligned}S(p,q) = Z[&I_{22},I_{23},I_{32},I_{33},I_{66},I_{67},I_{76},I_{77},p,I_{17},I_{25},\\
&q,I_{35},I_{38},I_{46},I_{47},I_{52},I_{53},I_{61},I_{64},I_{71},I_{74},I_{82},I_{83}]\end{aligned}$$

satisfies relations (7).

Lemma 4.2 is proved. ∎

Lemma 4.3. *Formula $I_{18}[(p \vee q)]$ is ultra-weak expressible in the logic $L\mathfrak{B}_2$ via formula F_3.*

Proof. Consider formula F_3. So, there are four ordered sets of elements $(\alpha_1,\ldots,\alpha_n)$, $(\beta_1,\ldots,,\beta_n)$, $(\gamma_1,\ldots,\gamma_n)$, $(\delta_1,\ldots,\delta_n)$ of the algebra $\mathfrak{B}_2$ such that

$$R_3(\alpha_i,\beta_i,\gamma_i,\delta_i),\ i=1,\ldots,n, \qquad (13)$$

and next relation is false

$$R_3(F_3[\alpha_1,\dots,\alpha_n],F_3[\beta_1,\dots,\beta_n],F_3[\gamma_1,\dots,\gamma_n],F_3[\delta_1,\dots,\delta_n]).$$

Consider new ordered sets of elements $(\alpha'_1,\dots,\alpha'_n)$, $(\beta'_1,\dots,\beta'_n)$, $(\gamma'_1,\dots,\gamma'_n)$, $(\delta'_1,\dots,\delta'_n)$ from algebra $\mathfrak{B}_2$ such that

$$\Delta\alpha_i=\Delta\alpha'_i,\ \Delta\beta_i=\Delta\beta'_i,\ \Delta\gamma_i=\Delta\gamma'_i,\ \Delta\delta_i=\Delta\delta'_i,$$
$$\alpha'_i\in\{\mathbf{0},\mathbf{1}\},\ \beta'_i\in\{\mathbf{0},\mathbf{1}\},\ \gamma'_i\in\{\mathbf{0},\mathbf{1}\},\ \delta'_i\in\{\mathbf{0},\mathbf{1}\}.$$

Clearly,

$$\begin{pmatrix}\alpha'_i\\ \beta'_i\\ \gamma'_i\\ \delta'_i\end{pmatrix}\subseteq\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}\tag{14}$$

Consider formula $F'_3=I_{18}[F_3]$. It is clear, by properties of F_3, we have

$$\begin{pmatrix}F'_3[\alpha_1,\dots,\alpha_n]\\ F'_3[\beta_1,\dots,\beta_n]\\ F'_3[\gamma_1,\dots,\gamma_n]\\ F'_3[\delta_1,\dots,\delta_n]\end{pmatrix}\subseteq\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{0}\end{pmatrix}\tag{15}$$

Examine formula $B(p_{\mathbf{1}},\dots,p_n)$ designed according to the scheme

$$B(p_1,\dots,p_n)=\begin{cases}F'_3, & \text{if } F'_3[\alpha'_1,\dots,\alpha'_n]=\mathbf{0},\\ I_{81}[F'_3], & \text{if } F'_3[\alpha'_1,\dots,\alpha'_n]=\mathbf{1}.\end{cases}$$

It is easy to verify that

$$\begin{pmatrix}B[\alpha_1,\dots,\alpha_n]\\ B[\beta_1,\dots,\beta_n]\\ B[\gamma_1,\dots,\gamma_n]\\ B[\delta_1,\dots,\delta_n]\end{pmatrix}\subseteq\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{1}\\ \mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}\tag{16}$$

Then build formula $B'(p,q,r)=B[B'_1,\dots,B'_n]$, where for any $i=1,\dots,n$

$$B'_i(p,q,r)=\mathbf{0},\ \text{if } \alpha'_i=\mathbf{0},\ \beta'_i=\mathbf{0},\ \gamma'_i=\mathbf{0},\ \delta'_i=\mathbf{0},$$

$$B'_i(p,q,r)=p,\ \text{if } \alpha'_i=\mathbf{0},\ \beta'_i=\mathbf{0},\ \gamma'_i=\mathbf{1},\ \delta'_i=\mathbf{1},$$
$$B'_i(p,q,r)=q,\ \text{if } \alpha'_i=\mathbf{0},\ \beta'_i=\mathbf{1},\ \gamma'_i=\mathbf{0},\ \delta'_i=\mathbf{1},$$
$$B'_i(p,q,r)=r,\ \text{if } \alpha'_i=\mathbf{0},\ \beta'_i=\mathbf{1},\ \gamma'_i=\mathbf{1},\ \delta'_i=\mathbf{0},$$
$$B'_i(p,q,r)=I_{81}[r],\ \text{if } \alpha'_i=\mathbf{1},\ \beta'_i=\mathbf{0},\ \gamma'_i=\mathbf{0},\ \delta'_i=\mathbf{1},$$
$$B'_i(p,q,r)=I_{81}[q],\ \text{if } \alpha'_i=\mathbf{1},\ \beta'_i=\mathbf{0},\ \gamma'_i=\mathbf{1},\ \delta'_i=\mathbf{0},$$

$$B_i'(p,q,r) = I_{81}(p),\ \text{if } \alpha_i' = \mathbf{1},\ \beta_i' = \mathbf{1},\ \gamma_i' = \mathbf{0},\ \delta_i' = \mathbf{0},$$

$$B_i'(p,q,r) = \mathbf{1},\ \text{if } \alpha_i' = \mathbf{1},\ \beta_i' = \mathbf{1},\ \gamma_i' = \mathbf{1},\ \delta_i' = \mathbf{1}$$

Therefore, according to relation (16), we obtain

$$\begin{pmatrix} B'[\mathbf{0},\mathbf{0},\mathbf{0}] \\ B'[\mathbf{0},\mathbf{1},\mathbf{1}] \\ B'[\mathbf{1},\mathbf{0},\mathbf{1}] \\ B'[\mathbf{1},\mathbf{1},\mathbf{0}] \end{pmatrix} \subseteq \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{17}$$

Let $\alpha = B'[\mathbf{0},\mathbf{0},\mathbf{0}]$, $\beta = B'[\mathbf{0},\mathbf{1},\mathbf{1}]$, $\gamma = B'[\mathbf{1},\mathbf{0},\mathbf{1}]$, $\delta = B'[\mathbf{1},\mathbf{1},\mathbf{0}]$. Consider formula $A(p,q,r)$ designed according to the scheme:

$$A(p,q,r) = \begin{cases} B'(p,q,r), & \text{if } \alpha = \mathbf{0},\ \beta = \mathbf{0},\ \gamma = \mathbf{0},\ \delta = \mathbf{1}, \\ B'(p, I_{81}[q], I_{81}[r]), & \text{if } \alpha = \mathbf{0},\ \beta = \mathbf{0},\ \gamma = \mathbf{1},\ \delta = \mathbf{0}, \\ B'(I_{81}(p), q, I_{81}[r]), & \text{if } \alpha = \mathbf{0},\ \beta = \mathbf{1},\ \gamma = \mathbf{0},\ \delta = \mathbf{0}, \\ B'(I_{81}(p), I_{81}[q], r), & \text{if } \alpha = \mathbf{0},\ \beta = \mathbf{1},\ \gamma = \mathbf{1},\ \delta = \mathbf{1}. \end{cases}$$

Formula $A(p,q,r)$ is ultra-weak expressible via formula F_3. It follows from (17) and from design of the formula $A(p,q,r)$ that

$$A[\mathbf{0},\mathbf{0},\mathbf{0}] = \mathbf{0},\ A[\mathbf{0},\mathbf{1},\mathbf{1}] = \mathbf{0},\ A[\mathbf{1},\mathbf{0},\mathbf{1}] = \mathbf{0},\ A[\mathbf{1},\mathbf{1},\mathbf{0}] = \mathbf{1}.$$

Therefore formula $I_{18}[(p \vee q)]$ is ultra-weak expressible via formula F_3 and $A(p,q,r)$ in each of the following sub-cases:

1) If $A[\mathbf{0},\mathbf{1},\mathbf{0}] = \mathbf{0}$, $A[\mathbf{1},\mathbf{0},\mathbf{0}] = \mathbf{0}$, then $I_{18}[(p \vee q)] = A[I_{18}(p), I_{18}[q], \mathbf{0}]$.
2) If $A[\mathbf{0},\mathbf{1},\mathbf{0}] = \mathbf{0}$, $A[\mathbf{1},\mathbf{0},\mathbf{0}] = \mathbf{1}$, then $I_{18}[(p \vee q)] = A[I_{81}(p), I_{18}(p), I_{18}[q]]$.
3) If $A[\mathbf{0},\mathbf{1},\mathbf{0}] = \mathbf{1}$, $A[\mathbf{1},\mathbf{0},\mathbf{0}] = \mathbf{0}$, then $I_{18}[(p \vee q)] = A[I_{18}(p), I_{81}(p), I_{18}[q]]$.
4) If $A[\mathbf{0},\mathbf{1},\mathbf{0}] = \mathbf{1}$, $A[\mathbf{1},\mathbf{0},\mathbf{0}] = \mathbf{1}$, then $I_{18}[(p \vee q)] = A[I_{81}(p), I_{18}[q], I_{81}[q]]$.

Lemma 4.3 is proved. ∎

Lemma 4.4. *Conjunction* $(p\&q)$ *is ultra-weak expressible in the logic* $L\mathfrak{B}_2$ *via formulas* $C(p,q)$ *and* $D(p,q)$ *satisfying conditions* (1)*, via formula* $S(p,q)$ *which respects restrictions* (7) *and via formula* $I_{18}[(p \vee q)]$.

Proof. Consider possible values for $S[\mathbf{1},\mathbf{0}]$ and design formulas $J'(p,q)$ and $J''(p,q)$ according to the schemes:

$$J'(p,q) = \begin{cases} I_{16}[S(p,q)], & \text{if } S[\mathbf{1},\mathbf{0}] \in \{\mathbf{0},\rho\}, \\ I_{71}[S(\neg q, \neg p], & \text{if } S[\mathbf{1},\mathbf{0}] \in \{\sigma,\mathbf{1}\}, \end{cases}$$

$$J''(p,q) = \begin{cases} I_{28}[S(p,q)], & \text{if } S[\mathbf{1},\mathbf{0}] \in \{\sigma,\mathbf{1}\}, \\ I_{83}[S(\neg q, \neg p], & \text{if } S[\mathbf{1},\mathbf{0}] \in \{\mathbf{0},\rho\}. \end{cases}$$

It is not so difficult to verify that

$$J'[\mathbf{0},\mathbf{0}] = \mathbf{0},\ J'[\sigma,\mathbf{1}] = \sigma,\ J'[\mathbf{1},\mathbf{1}] = \mathbf{1},$$

$$J''[\mathbf{0},\mathbf{0}] = \mathbf{0},\ J''[\mathbf{0},\rho] = \rho,\ J''[\mathbf{1},\mathbf{0}] = \mathbf{1}.$$

Now consider formulas

$$I_{56}(p) \vee I_{56}[q] = J'[I_{61}[C[I_{12}(p), I_{12}[q]]], I_{18}[(p \vee q)]],$$
$$I_{22}(p) \vee I_{22}[q] = J''[I_{18}[(p \vee q)], C[I_{22}(p), I_{22}[q]]].$$

It remains to verify that $(p\&q) = S[I_{56}(p) \vee I_{56}[q], I_{22}(p) \vee I_{22}[q]]$.
Lemma 4.4 is proved. ∎

Now the algorithm for detection whether a system of formulas is ultra-weak complete in the logic $L\mathfrak{B}_2$ is relatively simple. Suppose the system Σ of formulas is a list of formulas $G_1, \ldots, G_k$. To verify that it is ultra-weak complete in $L\mathfrak{B}_2$ it is sufficient to verify that for every class of formulas Π_i, $i = 1, 2, 3$ there is a formula G_{j_i}, $j_i \in \{1, \ldots, k\}$ that do not conserve the corresponding relation R_i $(i = 1, 2, 3)$ on the algebra $\mathfrak{B}_2$.

Theorem 4.2. *The propositional provability logic $L\mathfrak{B}_2$ is decidable relative to ultra-weak completeness of systems of formulas.*

Proof. It is obvious taking into account theorem 4.1 and the above described algorithm. ∎

5. CONCLUSIONS

Theorem 4.1 provide us necessary and sufficient conditions for detecting completeness of systems of formulas relative to ultra-weak expressibility in the propositional provability logic $L\mathfrak{B}_2$. We can consider a slice of extensions of GL [14], which also has an additional axiom $\Delta\Delta p$ and examine the conditions for completeness of formulas relative to ultra-weak expressibility in these logics. Note the logic $L\mathfrak{B}_2$ is an element of this slice of extensions. Also we can examine other types of expressibility (also weak or ultra-weak) of formulas: implicit expressibility, parametric expressibility, existential expressibility, etc.

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References

[1] E. L. Post, *Introduction to a general theory of elementary propositions*, Amer. J. Math., **43**(1921), p. 163 - 185.

[2] E. L. Post, *Two-valued iterative systems of mathematical logic*, Princeton University Press, Princeton, 1941.

[3] A. V. Kuznetsov, *Analogs of the 'Sheffer stroke' in constructive logic*, Sov. Math., Dokl., **160**, 2(1965), 274–277.

[4] A. V. Kuznetsov, *On functional expressibility in superintuitionistic logics*, Matematicheskie Issledovaniya, **6**, 4(1971), 75–122.

[5] M. F. Ratsa, *A criterion for functional completeness in the intuitionistic propositional logic*, Dokl. Akad. Nauk SSSR, **201**, 4(1971), 794–797.

[6] M. F. Ratsa, *Functional completeness in intuitionistic propositional logic*, Problemy Kibernet., **39**(1982), 107–150.

[7] M. F. Ratsa, *Expressibility in propositional calculi*, Ştiinţa, Chişinău, 1991.

[8] R. M. Solovay, *Provability interpretations of modal logic*, Israel J. Math., **25**, 3-4(1976), 287–304.

[9] R. Magari, *The diagonalizable algebras (the algebraization of the theories which express Theor.: II)*, Boll. Un. Mat. Ital. , **12**, 3(1975), 117–125.

[10] L. L. Maksimova, *Continuum of normal extensions of the modal logic of provability with the interpolation property* // Sib. Math. J. **30**, 6(1989), 935–944.

[11] A. V. Kuznetsov, *On detecting non-deducibility and non-expressibility* in: Logical deduction, Nauka, Moscow, 1979, 5–33.

[12] A. G. Russu, *A completeness criterion for expressibility in a four-valued extension of provability logic*, Bull. Acad. Sci. Mat., **20**, 1(1996), 75–86.

[13] A. Rusu, *On sufficient conditions for expressibility of constants in the 4-valued extension of the propositional provability logic GL*, Sem. Ser. in Math. and Comp. Sci., **1**, 1(2014), 29–40.

[14] W. J. Blok, *Pretabular varieties of modal algebras*, Studia Logica, **39**, 2-3(1980), 101–124.