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Superdense Coding of Information in Quantum Computer in the Paired Bosons Representation

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Abstract **— An alternative approach to superdense coding of information in quantum computing is proposed on the basis of Schwinger's two-boson representation of angular momentum. Since the effective spin S = 2^{n-1} - 1/2 corresponds to the n-qubit system, this representation can be used in the quantum computing. Operators of the logical elements of the quantum circuit were found, performing superdense coding of information in the paired bosons representation. It is shown that for superdense coding of information, the results obtained in the spinor representation and in the representation of paired bosons coincide. For one-qubit systems, one of the two representations cannot be favored. In the case of n-qubit systems for n >> 1, the representation of paired bosons is probably more convenient for applications, since in this representation the explicit form of the Pauli operators X, Y, and Z does not depend on n.**

Keywords — entanglement, Bose fields, CNOT gate, Hadamard gate, qubit, spinor operators

I. INTRODUCTION

Development of the information technologies have reached an effective methods of storing and processing of information. There is an essential difference between classical and quantum treatment of information, which became relevant after R. Feynman put forward the idea of simulating physics with computers [1]. Unlike the classical bit (0, 1), the qubit $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ ($|0\rangle$ and $|1\rangle$ denote basis vectors, α and β are complex numbers that satisfy the relation $|\alpha|^2+|\beta|^2=1$) geometrically is presented by the unit vector, which has its origin in the center of the Bloch sphere and end on the surface. There is an infinite set of points on the spherical surface and correspondingly any qubit may be in one from infinite set of states. It would seem that using only one qubit it is possible to store an infinite quantity of information. For this it is necessary to obtain a set of clones of one arbitrary state $|\psi\rangle$ and carry out measurements of the state of each copy, then with the probability $|\alpha|^2$ the qubit will be found in the state $|0\rangle$ and with the probability $|\beta|^2$ it will be found in the state $|1\rangle$. Since $|\alpha|^2$ and $|\beta|^2$ can have infinitely many values, it would be possible to obtain infinitely many information encoded in one qubit. However, it is impossible due to existence of no-cloning theorem of Wootters and Zurek [2].

Despite the restrictions caused by the no-cloning theorem, the number of publications in the field of quantum computing does not decrease, especially after the discovery of Deutsch–Jozsa [3], Shor [4] and Grover [5, 6] quantum algorithms. In all these investigations the spin algebra formalism is used in view of its simplicity. However, the Bose operators algebra is not more complicated then spinor algebra.

In this paper, we propose to implement the superdense coding of information using qubits and logical elements in the paired bosons (PB) representation. In Sect. II the logical elements given in the PB representation are found. The superdense coding of information in the paired bosons representation is theoretically studied in Sect. III. Concluding remarks are make in Sect. IV.

II. THE QUBIT BASIS VECTORS AND LOGICAL ELEMENTS IN PAIRED BOSONS REPRESENTATION

The basis vectors of any qubit traditionally are represented using the spin wave functions corresponding to an effective spin S = 1/2:

$$|0\rangle=|1/2,1/2\rangle, |1\rangle=|1/2,-1/2\rangle. \qquad (1)$$

Therefore all logical operations, including superdense coding of information, are given in terms of spinorial algebra. On the other hand, spin operators and spin wave functions can be defined in the Schwinger's paired bosons (PB) representation [7]. The superdense coding of information can be described in the PB representation, if needed for such investigation logical elements are given in the same representation. The rationale for this alternative approach is that in the spinor representation, the dimension of the matrices of spin operators increases with an increase in the number of qubits, while in the PB representation it does not depend on the number of qubits. In this case, all the specificity of multi-qubit system is

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contained in the structure of spin wave functions in the PB representation.

- *Basis vectors of the qubit*

In the PB representation, the components |0⟩ and |1⟩ of the qubit (1) has the form

$$\left|0\right\rangle = A^{+}\left|1\right\rangle_1\left|0\right\rangle_2, \left|1\right\rangle = B^{+}\left|0\right\rangle_1\left|1\right\rangle_2, \quad (2)$$

where A^+ and B^+ are the creation operators of Bose fields 1 and 2, which satisfy the relation

$$U_{1/2}^{+}U_{1/2} = A^{+}A + B^{+}B = I, \quad (3)$$

where I is an unit operator given in two- dimensional space.

- *Pauli X, Y, and Z gates*

Let $U_{1/2} = \begin{pmatrix} A \\ B \end{pmatrix}$ is an unitary spinor operator which satisfy the relation (3).

Let us subject each of the Pauli matrices σ_x, σ_y, and σ_z to a unitary transformation using the spinor operators $U_{1/2}^{+}$ and $U_{1/2}$:

$$\begin{aligned} U_{1/2}^{+}\sigma_x U_{1/2} &= A^{+}B + B^{+}A = X, \\ U_{1/2}^{+}\sigma_y U_{1/2} &= i(B^{+}A - A^{+}B) = Y, \\ U_{1/2}^{+}\sigma_z U_{1/2} &= A^{+}A - B^{+}B = Z \end{aligned} \quad (4)$$

Thus, X, Y, and Z are Pauli operators in the paired bosons representation. In the spinor representation Pauli operators are unitary operators. Therefore, in the representation of paired bosons, they are also unitary.

The eigenvectors of operators X, Y, and Z are $\frac{1}{\sqrt{2}}\bullet$ $(|1\rangle_1|0\rangle_2 \pm |0\rangle_1|1\rangle_2)$ for X, $\frac{1}{\sqrt{2}}\bullet$ $(|1\rangle_1|0\rangle_2 \pm i|0\rangle_1|1\rangle_2)$ for Y, and correspondingly $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$ for Z operator.

Unlike a one-qubit system, a system of n qubits is characterized by an effective spin $S = 2^{n-1}-1/2$ [8]. In this case, the relationship (3) takes the form

$$U_S^{+}U_S = A^{+}A + B^{+}B = 2SI, \quad (5)$$

where I is the unit operator given in ($2S$+1) - dimensional space.

- *Hadamard logical element H*

Let us act by the operator H on the input qubit vector

$$\left|\psi\right\rangle = \alpha\left|1\right\rangle_1\left|0\right\rangle_2 + \beta\left|0\right\rangle_1\left|1\right\rangle_2, \quad (6)$$

where α and β, as stated in Section I, are complex numbers that satisfy the condition

$$|\alpha|^2 + |\beta|^2 = 1. \quad (7)$$

As a result of this action, we obtain

$$\begin{aligned} &H\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left[A^{+}(A+B) + B^{+}(A-B)\right] \\ &\times(\alpha\left|1\right\rangle_1\left|0\right\rangle_2 + \beta\left|0\right\rangle_1\left|1\right\rangle_2) = \\ &\frac{1}{\sqrt{2}}\left[(\alpha+\beta)\left|1\right\rangle_1\left|0\right\rangle_2 + (\alpha-b)\left|0\right\rangle_1\left|1\right\rangle_2\right]. \end{aligned} \quad (8)$$

The Hadamard operator H from (8) is a unitary operator since

$$\begin{aligned} &H = \frac{1}{\sqrt{2}}(X+Z), H^{+} = H, \\ &H^{+}H = (1/2)(X^2 + Z^2 + iY - iY) = I \end{aligned} \quad (9)$$

The normalized eigenvectors of the operator H were calculated in [9] in the spinor representation:

$$\left|\psi_1\right\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ a \end{pmatrix}, \left|\psi_2\right\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ b \end{pmatrix}, \quad (10)$$

where $a = \sqrt{2}-1, b = -\sqrt{2}-1$.

Moving on from the spinor representation to the paired bosons representation, we obtain

$$\left|\psi_1\right\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\left[\left|1\right\rangle_1\left|0\right\rangle_2 + (\sqrt{2}-1)\times\left|0\right\rangle_1\left|1\right\rangle_2\right], \quad (11)$$

$$\left|\psi_2\right\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\left[\left|1\right\rangle_1\left|0\right\rangle_2 - (\sqrt{2}+1)\times\left|0\right\rangle_1\left|1\right\rangle_2\right]. \quad (12)$$

Formulas (4) and (8) define the operators of the logical elements X, Y, Z, and H in the paired bosons representation.

III. Superdense Coding of Information in Paired Bosons Representation

Superdense coding is a quantum protocol, which allows increasing the information content using such a key resource of quantum systems as entanglement. The central idea is that two bits of classical information can be transferred with a single qubit participating in communication [10].

Let us imagine that Alice intends to send two classical bits of information to Bob using qubits. Without relying on entanglement, Alice has to send two qubits in order to transfer two bits of information. However, the advantage appears provided Bob prepares two entangled qubits and sends only one of them to Alice. Alice encodes information in this qubit by applying one single- and one two-qubit gates (Hadamard- and CNOT-gates) and this qubit is then sent back to Bob. Bob performs Bell measurements of both qubits and extracts two classical

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bits of information despite of the fact that only the single qubit has been utilized in quantum communication.
It should be noted that the four entangled quantum states, discussed for the first time by Einstein, Podolsky and Rosen [11], are called EPR states. The properties of these states were studied by Bell [12], in connection with which they are also called Bell states.

On the first step two qubits are prepared by the Bob in an entangled states. He initially start the two qubits in the basis state $|1\rangle_1|0\rangle_2$. He applied the Hadamard gate (H) to the first qubit to create superposition of the states $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$. He then applied CNOT gate U_{CN} using the first qubit as a control and the second as a target (Fig. 1).

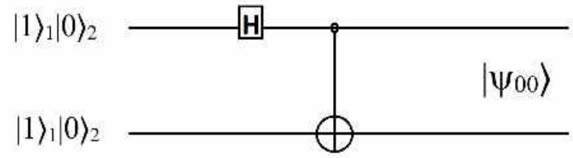

Figure 1. Quantum scheme described by the operator L = I ⊗ H • CNOT for obtaining the Bell state $|\psi_{00}\rangle$.

At the output of the quantum circuit shown in Fig. 1, after the original signal passes through the Hadamard gate H, and then through the CNOT gate, the entangled state is formed (the Bell or EPR state $|\psi_{00}\rangle$):

$$\left|\psi_{00}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|3\right\rangle_1\left|0\right\rangle_2 + \left|0\right\rangle_1\left|3\right\rangle_2\right). \quad (13)$$

Really, let the basis vectors $|1\rangle_1|0\rangle_2$ be fed to both inputs of the quantum circuit. Let us find the state vector after passing through the Hadamard gate:

$$H\left|1\right\rangle_1\left|0\right\rangle_2 = \frac{1}{\sqrt{2}}\left(X+Z\right)\left|1\right\rangle_1\left|0\right\rangle_2 = \frac{1}{\sqrt{2}}\left(\left|1\right\rangle_1\left|0\right\rangle_2 + \left|0\right\rangle_1\left|1\right\rangle_2\right) \quad (14)$$

where X and Z are given in (4). We got a vector coming to the control input of the CNOT element. The base vector arrives at the controlled input of the CNOT element is $|1\rangle_1|0\rangle_2$. Thus, the input of the CNOT element receives the vector $|\psi\rangle$, which in the spinor representation has the form [3]

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)\otimes\left|0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\otimes\left|0\right\rangle + \left|1\right\rangle\otimes\left|0\right\rangle\right) \quad (15)$$

In the paired bosons representation the vector $|\psi\rangle$ is converted to the form

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left(\left|3\right\rangle_1\left|0\right\rangle_2 + \left|1\right\rangle_1\left|2\right\rangle_2\right). \quad (16)$$

The action of the CNOT operator on such an input vector results in the Bell sate $|\psi_{00}\rangle$ from (19):

$$U_{CN}\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}\left(\left|3\right\rangle_1\left|0\right\rangle_2 + \left|1\right\rangle_1\left|2\right\rangle_2\right) = \frac{1}{\sqrt{2}}\left(\left|3\right\rangle_1\left|0\right\rangle_2 + \left|0\right\rangle_1\left|3\right\rangle_2\right) \quad (17)$$

If basis vectors $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$ are applied to the control and controlled inputs of the quantum circuit, then at the output of the circuit, after passing through the Hadamard and CNOT gates, another Bell state will be formed:

$$\left|\psi_{01}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|1\right\rangle_1\left|2\right\rangle_2 + \left|2\right\rangle_1\left|1\right\rangle_2\right). \quad (18)$$

In a similar way, one can obtain the remaining two Bell states $|\psi_{10}\rangle$ and $|\psi_{11}\rangle$, if in the first case the basis vectors $|0\rangle_1|1\rangle_2$ and $|1\rangle_1|0\rangle_2$ are fed to the control and controlled inputs of the quantum circuit, and in the second case two $|0\rangle_1|1\rangle_2$ are fed to both inputs:

$$\left|\psi_{10}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|3\right\rangle_1\left|0\right\rangle_2 - \left|0\right\rangle_1\left|3\right\rangle_2\right), \quad (19)$$

$$\left|\psi_{11}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|2\right\rangle_1\left|1\right\rangle_2 - \left|1\right\rangle_1\left|2\right\rangle_2\right). \quad (20)$$

All four cases (13), (18) - (20) can be written briefly in the form

$$\left|\psi_{ab}\right\rangle = \left(I\otimes H\right)\cdot CNOT\left|ab\right\rangle; a,b = 0,1. \quad (21)$$

The end goal is for Alice to send two classical bits of information to Bob using one qubit. But before she does that, she needs to apply a set of quantum gates to her qubit, depending on which two bits of information she is going to send. Alice encodes each pair of consecutive bits ab using the L_{ab} operator and acts on her pair qubit. She matches pairs 00, 01, 10, and 11 with the operators L_{00} =I, L_{01} = X, L_{10} = Z, and L_{11} =*i*Y, where I, X, Z, and Y are

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defined by (3) and (4). Alice can only affect her pair qubit. Therefore, Alice's influence on any pair ab is described by the operator $L_{ab} \otimes I$. Taking into account (13), the following relations can be obtained:

$$\begin{aligned}
(L_{00}\otimes I)|\psi_{00}\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle_1|0\rangle_2+|0\rangle_1|3\rangle_2\right)=|\psi_{00}\rangle,\\
(L_{01}\otimes I)|\psi_{00}\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle_1|1\rangle_2+|1\rangle_1|2\rangle_2\right)=|\psi_{01}\rangle,\\
(L_{10}\otimes I)|\psi_{00}\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle_1|0\rangle_2-|0\rangle_1|3\rangle_2\right)=|\psi_{10}\rangle,\\
(L_{11}\otimes I)|\psi_{00}\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle_1|1\rangle_2-|1\rangle_1|2\rangle_2\right)=|\psi_{11}\rangle
\end{aligned} \quad (22)$$

Thus,

$$(L_{ab}\otimes I)|\psi_{00}\rangle = |\psi_{ab}\rangle; a,b=0,1. \quad (23)$$

After encoding the information, Alice sends her qubit to Bob. If she sends Bob a sequence of *ab* qubits, then Bob has an entangled pair of qubits in the Bell state $|\psi_{ab}\rangle$. To decode this signal, it is necessary to apply a quantum circuit presented on the **Figure 2.**

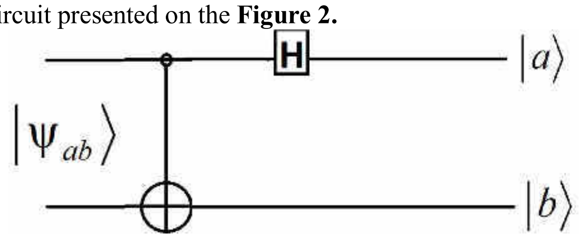

Figure 2. Deconding scheme of the method of superdense coding of information

The action of the decoding operator on the entangled Bell state $|\psi_{ab}\rangle$ leads to the result. The action of the decoding operator on the entangled Bell state $|\psi_{ab}\rangle$ leads to the result

$$((I\otimes H)\cdot CNOT)^{-1}|\psi_{ab}\rangle = |a\rangle\otimes|b\rangle = |ab\rangle. \quad (24)$$

Taking into account the unitarity properties of the operators $I\otimes H$ and CNOT, we finally find:

$$\begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix}\cdot(I\otimes H)|\psi_{ab}\rangle = |ab\rangle. \quad (25)$$

CONCLUSIONS

1. Traditionally, in quantum informatics the qubits and logical elements are treated in terms of spinor algebra. Along with this, quantum computing can be performed in the paired bosons representation.
2. Basic gates were found in the paired bosons representation.
3. Superdense coding of information in the representation of paired bosons was performed.
4. Superdense coding of information is closely related with quantum teleportation. To avoid confusion we need to clarify the difference. Superdense coding is a procedure that allows someone to send two classical bits to another party using just a single qubit of communication. Quantum teleportation is a process by which the state of qubit ($|\psi\rangle$) can be transferred from one location to another, using two bits of classical communication and a Bell pair. We can say that teleportation is a process that destroys the quantum state of a qubit in one location and recreate it on a qubit at distant location. Thus, the teleportation protocol is a flipped version of the superdense coding protocol.

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