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Graphical methods as a complements of analytical methods used in the research of dynamic models for networks reliability

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***Abstract*—Our work deals with a typical problem of comparing the reliability of a serial-parallel type network vs the reliability of a parallel-serial type network. Using graphic methods on elementary models, we show how they lead to the formulation of mathematically argued conclusions. These conclusions are then extended to whole families of probabilistic dynamic models related to the initial models.**

Keywords—lifetime distribution, survivival / reliability function, serial-parallel and parallel-serial networks

I. INTRODUCTION

Our problem appears in the context of studying dynamic probabilistic models, looking at identifying the conditions in which a serial-parallel network is always more reliable than a parallel-serial network. We will rely, essentially, on the fact that for networks of this type in which the number of subnets, but also the number of elements are not random variables, in the paper [1] the analytical formulas for calculating the corresponding survival/reliability functions were deduced.

We remind you that, according to the generally accepted definition in the specialized literature, by the survival / reliability function of a system (or networks) we will understand the function $R(x)$ which coincides with the probability that the lifetime of this system will exceed the time threshold x, i.e., $R(x)=1-F(x)$, where $F(x)$ is lifetime cummulative distribution function (c.d.f.) of the sistem.

Thus, in the papers [1]-[2], thanks to the calculation formulas, the sufficient condition was found that a network of type ***A***, i.e. of serial-parallel type, is always more reliable than a network of type ***B***, i.e. of parallel-serial type. We specify that the nominated networks have the same number of subnets equal to M, $M\in\{2,3,\ldots\}$, the number of elements in each subnet being equal to Ni, $i=1,M$, and the lifetimes of all elements are nonnegative independent, identically distributed random variables, (i.i.d.r.v.). We notice that the case $M=1$ was omitted, because the network of type ***A*** becomes, in this way, the network of parallel type, and the network of type ***B*** becomes the network of serial type, of which it is well known that the first network is, always, more reliable than the second one.

The formulas derived in the paper [1] allowed us to extend to the case of dynamic modeling the conclusion made in [3]-[4] for the static modeling regarding the reliability of the ***C***-type parallel-serial network in which the number of subnets is equal to N, $N\geq 2$ and the number of elements in the each subnet is the same and equal to M versus the reliability of the ***D***-type network in which the number of subnets, on the contrary, is equal to M, $M \geq 2$, the number of elements in each subnet being equal to N , and the lifetimes of all elements being v.a.i.i.d. More precisely, in paper [1] it was demonstrated that in the dynamic case, like in the static case, the ***C***-type network is always more reliable than the ***D***-type network, regardless of the values of M and N and regardless of lifetime c.d.f. $F(x)$ of each element. The question arises, but what will happens if the number of elements in each of the N subnets of the ***C***-type network can be different from M, and the number of elements in each of the M subnets of the ***D***-type network can, analogously, be different from N. Next, the duestion arises: will you still have Type ***C*** network (series-parallel) more reliable than type ***D*** network (parallel-series) or not? If yes, under what conditions? These are the questions we aim to answer.

II. Reliability of modified ***C*** network vs modified ***D*** network reliability

We will call, in the following, the models described above the modified ***C*** and ***D*** networks.

If we denote by $R_{s-p}(x)$ and by $R_{p-s}(x)$, respectively, the reliability functions of the modified networks of type ***C*** and ***D***, then, according to the formulas derived in [1], we will have that

$$R_{s-p}(x) = \prod_{k=1}^{N}\left[\left(1-(F(x))^{M_k}\right]\right. \quad \textbf{(1)}$$

$$R_{p-s}(x) = 1 - \prod_{k=1}^{M}\left[1-\left(1-(F(x))^{N_k}\right)\right] \quad \textbf{(2).}$$

Question: what can we say, based on these analytical formulas, about the reliability of one network vs the reliability of the other? If, in particular case, $M_1 = M_2 = \ldots = M_N = M$ *and* $N_1 = N_2 = \ldots = N_M = N$, then we know, due to the paper [1], that the reliability of the modified Type ***C*** network is always higher than the reliability of the modified Type ***D*** network, otherwise the answer is not clear. In order to identify a more general rule, we will first use the method of graphical representation of the reliability of both networks according to particular cases. But we mention that, as was shown in the paper [1], this positioning does not depend on the lifetime c.d.f. of each element of the network. So, Therefore, in all the following examples, we can consider that the lifetime X of each unit has uniform c.d.f. , i.e. $F(x)=x$, for $x\in[0,1]$, otherwise $F(x)=0$, if $x<0$ and $F(x)=1$, if $x>1$, shortly $X \sim U[0,1]$. The case M = N being analyzed in the same paper [1], we will analyze separately the cases $M > N$ and $M < N$ under what conditions $M_1 = M_2 = \ldots = M_N = M$ *and* $N_1 = N_2 = \ldots = N_M = N$ are not fulfilled. Reliability function $R_{s-p}(x)$ will be represented graphically with a ***continuous line*** and the reliability function $R_{p-s}(x)$ will be represented with a ***broken line***.

Example 1. a) $X \sim U[0,1]$, $M>N$; $M=3$, $N=2$; $M_1 = M_2 = 3$; $N_1 = 1$, $N_2 = 3$, $N_3 = 2$; $R_{s-p}(x)= (1-x^3)^2$, $R_{p-s}(x)= 1-(1-(1-x))\,(1-(1-x)^3)\,(1-(1-x)^2)$ (see Fig.1.a);

b) $X \sim U[0,1]$, $M>N$; $M=3$, $N=2$; $M_1 = 2$ $M_2 = 4$; $N_1 = 1$, $N_2 = 1$, $N_3 = 4$; $R_{s-p}(x)= (1-x^2)\,(1-x^4)$, $R_{p-s}(x)= 1-(1-(1-x))^2\,(1-(1-x)^4)$ (see Fig.1.b);

c) $X \sim U[0,1]$, $M>N$; $M=3$, $N=2$; $M_1 = 1$ $M_2 = 5$; $N_1 = N_2 = N_3 = 2$; $R_{s-p}(x)= (1-x)\,(1-x^5)$, $R_{p-s}(x)= 1-(1-(1-x)^2)^3$ (see Fig.1.c);

d) $X \sim U[0,1]$, $M>N$; $M=3$, $N=2$; $M_1 = 1$ $M_2 = 5$; $N_1 = 1$, $N_2 = 3$, $N_3 = 2$; $R_{s-p}(x)= (1-x)\,(1-x^5)$, $R_{p-s}(x)= 1-(1-(1-x))\,(1-(1-x)^3)\,(1-(1-x)^2)$ (see Fig.1.d);

e) $X \sim U[0,1]$, $M>N$; $M=3$, $N=2$; $M_1 = M_2 = 3$; $N_1 = N_2 = N_3 = 2$; $R_{s-p}(x)= (1-x^3)^2$, $R_{p-s}(x)= 1-(1-(1-x)^2)^3$ (see Fig.1.e);

f) $X \sim U[0,1]$, $M>N$; $M=3$, $N=2$; $M_1 = 3$ $M_2 = 5$; $N_1 = 2$, $N_2 = 3$, $N_3 = 4$; $R_{s-p}(x)= (1-x^3)\,(1-x^5)$, $R_{p-s}(x)= 1-(1-(1-x)^2)\,(1-(1-x)^3)\,(1-(1-x)^4)$ (see Fig.1.f).

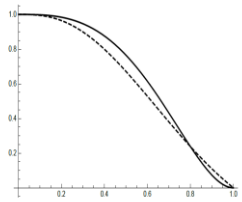
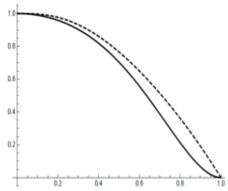

Fig. 1.a. Unclair situation. Fig. 1.b $R_{s-p}(x) \le R_{p-s}(x)$.

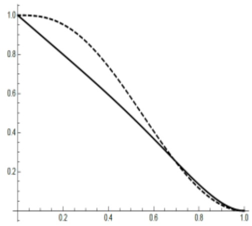
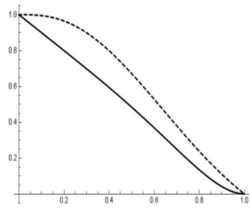

Fig. 1.c. Unclair situation. Fig. 1.d. $R_{s-p}(x) \le R_{p-s}(x)$.

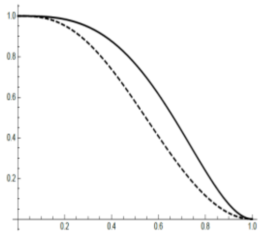
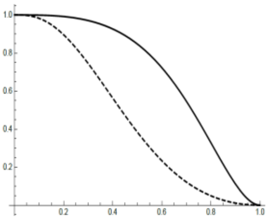

Fig. 1.e. $R_{s-p}(x) \ge R_{p-s}(x)$. Fig. 1.f. $R_{s-p}(x) \ge R_{p-s}(x)$.

Now we will give a similar example for the case M<N.

Example 2. a) $X \sim U[0,1]$, $M<N$; $M=2$, $N=3$; $M_1 = M_2 = M_3 = 2$; $N_1 = 1$, $N_2 = 5$; $R_{s-p}(x)= (1-x^2)^3$, $R_{p-s}(x)= 1-(1-(1-x))\,(1-(1-x)^5)$ (see Fig.2.a);

b) $X \sim U[0,1]$, $M<N$; $M = 2$, $N=3$; $M_1 = 1$, $M_2 = 3$, $M_3 = 2$; $N_1 = 1$, $N_2 = 5$; $R_{s-p}(x)= (1-x)(1-x^3)\,(1-x^2)$, $R_{p-s}(x)= 1-(1-(1-x))\,(1-(1-x)^5)$ (see Fig.2.b);

c) $X \sim U[0,1]$, $M<N$; $M=2$, $N=3$; $M_1 = 1$, $M_2 = 1$, $M_3 = 4$; $N_1 = N_2 = 3$; $R_{s-p}(x)= (1-x)^2 (1-x^4)$, $R_{p-s}(x)= 1-(1-(1-x)^3)^2$ (see Fig.2.c);

d) $X \sim U[0,1]$, $M<N$; $M=2$, $N=3$; $M_1 = 1$, $M_2 = 1$, $M_3 = 4$; $N_1 = 1$, $N_2 = 5$; $R_{s-p}(x)= (1-x)^2 (1-x^4)$, $R_{p-s}(x)= 1-(1-(1-x)) (1-(1-x)^5$ (see Fig.2.d);

e) $X \sim U[0,1]$, $M<N$; $M=2$, $N=3$; $M_1 = M_2 = M_3 = 2$; $N_1 = N_2 = 3$; $R_{s-p}(x)= (1-x^2)^3$, $R_{p-s}(x)= 1-(1-(1-x)^3)^2$ (see Fig.2.e);

f) $X \sim U[0,1]$, $M<N$; $M=2$, $N=3$; $M_1 = 3$ $M_2 = 5$, $M_3 = 2$; $N_1 = 3$, $N_2 = 4$; $R_{s-p}(x)= (1-x^2) (1-x^3) (1-x^5)$, $R_{p-s}(x)= 1-(1-(1-x)^3) (1-(1-x)^4)$ (see Fig.2.f).

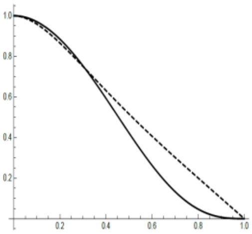
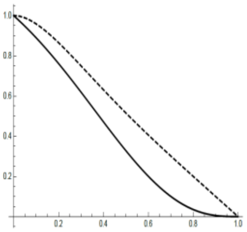

Fig. 2.a. Unclair situation. Fiig. 2.b. $R_{s-p}(x) \leq R_{p-s}(x)$.

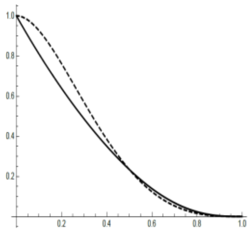
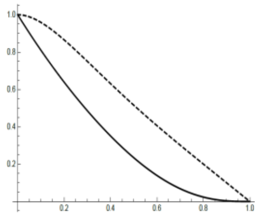

Fig. 2.c. Unclair situation. Fig. 2.d. $R_{s-p}(x) \leq R_{p-s}(x)$.

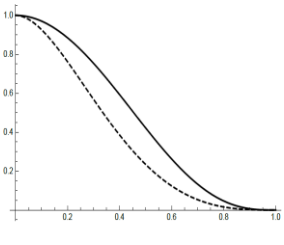
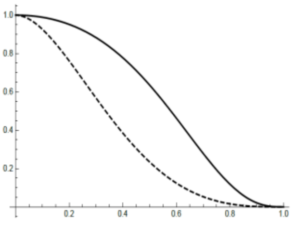

Fig. 2.e. $R_{s-p}(x) \geq R_{p-s}(x)$. Fig. 2.f. $R_{s-p}(x) \geq R_{p-s}(x)$.

Next we will consider that lifetime X of each units is no longer uniform distributed r.v. as in examples 1-2. So, let us consider, as another example, that c.d.f. of r.v. X coincides with exponential distribution with parameter $\lambda=1$, i.e., $F(x)=1-exp\{-x\}$, for $x\geq 0$ and $F(x)=0$, for $x<0$, shortly $X \sim exp\{1\}$.

Example 3. a) $X \sim exp\{1\}$, $M>N$; $M=3$, $N=2$; $M_1 = M_2 = 3$; $N_1 = 1$, $N_2 = 3$, $N_3 = 2$; $R_{s-p}(x)= (1-(1-e^{-x})^3)^2$, $R_{p-s}(x)= 1-(1-(1-(1-e^{-x}))) (1-(1-(1-e^{-x}))^3) (1-(1-(1-e^{-x}))^2)$ (see Fig.3.a);

b) $X \sim exp\{1\}$, $M>N$; $M=3$, $N=2$; $M_1 = M_2 = 3$; $N_1 = N_2 = N_3 = 2$; $R_{s-p}(x)= (1-(1-e^{-x})^3)^2$, $R_{p-s}(x)= 1-(1-(1-(1-e^{-x}))^2)^3$ (see Fig.3.b);

c) $X \sim exp\{1\}$, $M>N$; $M=3$, $N=2$; $M_1 = 1$ $M_2 = 5$; $N_1 = 1$, $N_2 = 3$, $N_3 = 2$; $R_{s-p}(x)= (1-(1-e^{-x})) (1-(1-e^{-x})^5)$, $R_{p-s}(x)= 1-(1-(1-(1-e^{-x}))) (1-(1-(1-e^{-x}))^3) (1-(1-(1-e^{-x}))^2)$ (see Fig.3.c);

d) $X \sim exp\{1\}$, $M<N$; $M=2$, $N=3$; $M_1 = 3$ $M_2 = 5$, $M_3 = 2$; $N_1 = 3$, $N_2 = 4$; $R_{s-p}(x)= (1-x^2) (1-x^3) (1-x^5)$, $R_{p-s}(x)= 1-(1-(1-x)^3) (1-(1-x)^4)$ (see Fig.3.d).

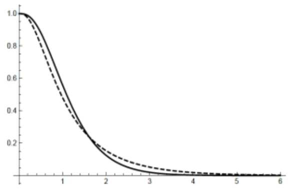
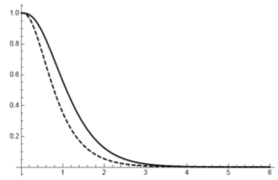

Fig. 3.a. Unclair situation. Fig. 3.b. $R_{s-p}(x) \geq R_{p-s}(x)$.

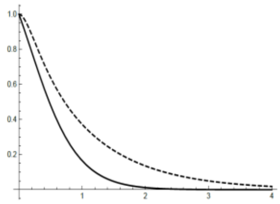
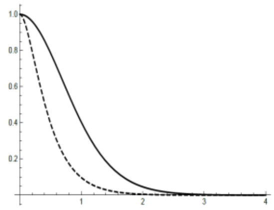

Fig. 3.c. $R_{s-p}(x) \leq R_{p-s}(x)$. Fig. 3.d. $R_{s-p}(x) \geq R_{p-s}(x)$.

III. CONCLUSIONS BASED ON THE GRAPHIC EXAMPLES

From examples 1.a) - 1.d) and 2.a) - 2.d) given previously we can draw the following empirical conclusions: a) Regardless of the fact that $M<N$ or $M>N$, when $min(M_1, M_2,...., M_N)<M$ or $min(N_1, N_2,...,N_M)<N$, we cannot say anything with certainty about the reliability of Type ***C*** network (serial-parallel) vs the reliability of Type ***D*** network (parallel-serial). On the contrary, the situation changes radically in examples 1.e) - 1.f) and 2.e - 2f, regardless of the fact that $M<N$ or $M>N$, in the sense that, when the conditions $min(M_1,M_2,....,M_N)\geq M$ and $min(N_1, N_2,...,N_M)\geq N$ are satisfied, we can say with certainty that networks of type ***C*** (serial-parallel) will be more reliable than networks of type ***D*** (parallel-serial).

Example 3 graphically show us that conclusions made above a valid regardless of lifetime c.d.f. $F(x)$.

The result of these graphic experiments suggests, in fact, that the following statement becomes plausible, but which must be proved mathematically.

Propozition. *If the numbers $M_1, M_2,..., M_N$ of the units included in each of the N subnets of the modified **C**-type network and the numbers $N_1, N_2,..., N_M$ of the units included in each of the M subnets of the modified **D**-type network (simultaneously) satisfy the conditions $min(M_1, M_2,...., M_N)\geq M$ and $min(N_1, N_2,...,N_M)\geq N$, then, regardless of lifetime c.d.f. F(x) of each units, the **C**-type network is more reliable than the **D**-type network.*

Prove. In fact, to prove our statement it is enough to show, according to formulas (1) and (2), that, regardless of lifetime c.d.f. F(x) of each units,

$$R_{s-p}(x) = \prod_{k=1}^{N}\left[\left(1-(F(x))^{M_k}\right]\geq\right.$$

$$\geq R_{p-s}(x) = 1-\prod_{k=1}^{M}\left[\left(1-(1-F(x))^{N_k}\right].\right.$$

But in the paper *[1]* it was shown that, due to the characteristic properties of the c.d.f., this kind of inequalities, if they occur, they are valid for any lifetime c.d.f. *F(x)*. Therefore, if we want to prove this inequality, it is enough to show that it is valid for the case when F(x) coincides with the uniform distribution on the interval *[0,1]*. So, we have to prove that

$$\prod_{k=1}^{N}[1-x^{M_k}]\leq 1-\prod_{k=1}^{M}[(1-(1-x)^{N_k}].$$

Another words, we have to prove that

$$\prod_{k=1}^{N}[1-x^{M_k}]+\prod_{k=1}^{M}[(1-(1-x)^{N_k}]\geq 1$$

for every $x\in[0,1]$. On the other hand, due to the conditions $min(M_1, M_2,...., M_N)\geq M$ and $min(N_1, N_2, ..., N_M)\geq N$, we have that for every $x\in[0,1]$

$$\prod_{k=1}^{N}[1-x^{M_k}]+\prod_{k=1}^{M}[(1-(1-x)^{N_k}]\geq$$

$$\geq (1-x^M)^N+(1-x^N)^M.$$

Because, from paper [1] , we know that

$(1-x^M)^N+(1-x^N)^M\geq 1$, for each $M,N\in\{1, 2, 3, ...\}$ this fact completes the proof of the statement from our Proposition.

The final conclusion resides in the folowing: the graphic method applied in Network's reliability is thery useful not only for visualizing reliabilty of the ghiven system, but also for identifying some mathematical laws regarding the reliability of one network vs another Network.

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