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Application of the Schwinger's oscillator model of angular momentum to quantum computing

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Abstract— **The Schwinger's oscillator model of angular momentum is applied to define quantum logical elements in quantum circuits by means of wave functions of two independent harmonic oscillators. It is shown how four EPR entangled states can be determined based on this model.**

Keywords—qubit; quantum computing; logical elements; quantum circuit

I. Introduction

Unlike calculations in classical computer science, based on the concept of a bit, quantum computer science uses the concept of a quantum bit (qubit). A qubit is a quantum system that can be in two states |0⟩ and |1⟩ (the Dirac notations [1] are used). An example of such systems is an electron with two spin orientations or a photon with two possible polarizations. In the general case, the state of a qubit is given by a superposition of basis vectors |0⟩ and |1⟩ with coefficients α and β satisfying the condition $|\alpha|^2+|\beta|^2 = 1$, which follows from the normalization condition for the qubit wave function. The fundamental difference between a qubit and a bit is that a qubit can be simultaneously in states |0⟩ and |1⟩ (the coefficients α and β are simultaneously different from zero). This actually determines the difference in the ways of constructing classical and quantum computers. We note that at present quantum computers are not yet mass-produced, but experimental samples have already been built (see, for example, the results of calculations carried out on one of them [2]).

To describe multi-qubit systems, it is convenient to introduce the effective spin $S = 2^{(N-1)}-1/2$ [3], where N is the number of qubits. The N-qubit system is characterized by 2S+1 states |S, S⟩, |S, S-1⟩, |S, S-2⟩, …, |S, 2-S⟩, |S, 1-S⟩, |S, -S⟩. Any N-qubit system can be in any of these states, as well as in any state of their superposition.

The Schwinger oscillator model of angular momentum [4] can be used to conduct quantum computing [3]. In this case, it is necessary to convert single- and multi-qubit logical elements from the spinor representation to the two-boson one. This article discusses some of the features of applying the two-boson Schwinger representation of the effective spin to quantum computing

II. On Two-Bosonic Schwinger Representation of Angular Momentum

The two-boson Schwinger representation for the effective spin is realized using two types of independent harmonic oscillators. We denote by A_1^+, A_1 and A_2^+, A_2 the Bose operators of creation and annihilation corresponding to these harmonic oscillators. Then, in the system of units in which Planck's constant ħ=1, the spin projection operators S_x, S_y, and S_z are defined by the expressions [4]

$$S_x = \frac{1}{2}\left(A_1^+A_2 + A_2^+A_1\right), S_y = \frac{1}{2i}\left(A_1^+A_2 - A_2^+A_1\right), \quad S_z = \frac{1}{2}\left(A_1^+A_1 - A_2^+A_2\right) \tag{1}$$

The spin wave function in the two-boson Schwinger representation of the angular momentum has the form

$$|S,M\rangle = \frac{1}{\left[(S+M)!(S-M)!\right]^{1/2}}\left(A_1^+\right)^{S+M}\left(A_2^+\right)^{S-M}|0\rangle \tag{2}$$

where |0⟩ denotes the vacuum state $|0\rangle = |0\rangle_1|0\rangle_2$.

III. Logical Elements of a One-Qubit System in a Two-Boson Representation of the Effective Spin S=1/2

A. Logical element NOT (or Pauli-X)

In the two-boson representation of the effective spin S = 1/2, the Pauli-X logical element of the one-qubit system has the form

$$X = \left(A_1^+ A_2^+\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = A_1^+A_2 + A_2^+A_1. \tag{3}$$

Let us act by operator *X* on the wave function of the qubit $|\psi\rangle = \alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2$:

$$X|\psi\rangle = \left(A_1^+A_2 + A_2^+A_1\right)(\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2) = \beta|1\rangle_1|0\rangle_2 + \alpha|0\rangle_1|1\rangle_2 \quad (4)$$

It can be seen that as a result of such an action, the coefficients α and β in the original qubit are reversed (that is, $\alpha \to \beta$ and $\beta \to \alpha$).

The normalized eigenvectors of operator X in the representation of paired bosons have the form

$$|\psi\rangle_{1,2} = \frac{1}{\sqrt{2}}\left(|1\rangle_1|0\rangle_2 \pm |0\rangle_1|1\rangle_2\right). \quad (5)$$

B. Logical element Z (or Pauli-Z)

In the two-boson representation of the effective spin S = 1/2, the Pauli-X logical element of the one-qubit system has the form

$$Z = \left(A_1^+A_2^+\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = \left(A_1^+A_1 - A_2^+A_2\right). \quad (6)$$

Under action of operator Z on the wave function of a qubit $|\psi\rangle = \alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2$, we obtain

$$Z|\Psi\rangle = \left(A_1^+A_1 - A_2^+A_2\right)(\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2) = (\alpha|1\rangle_1|0\rangle_2 - \beta|0\rangle_1|1\rangle_2). \quad (7)$$

As can be seen from (7), the operator Z does not change the coefficient at the basis vector $|1\rangle_1|0\rangle_2$ and changes the sign of the coefficient at the basis vector $|0\rangle_1|1\rangle_2$ .The eigenvectors of the operator Z coincide with the eigenvectors $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$.

C. Hadamard gate H

In the spinor basis, the Hadamard logical element is given by the matrix [3]

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \quad (8)$$

In the two-boson representation, the operator H has the form

$$H = \frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\left[A_1^+(A_1+A_2) + A_2^+(A_1-A_2)\right], \quad (9)$$

where X and Z are real Pauli matrices.

The action of the operator H on the wave function of the qubit |Ψ⟩ leads to the following result:

$$\frac{1}{\sqrt{2}}\left[A_1^+(A_1+A_2) + A_2^+(A_1-A_2)\right](\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2) = \frac{1}{\sqrt{2}}\left[(\alpha+\beta)|1\rangle_1|0\rangle_2 + (\alpha-b)|0\rangle_1|1\rangle_2\right]. \quad (10)$$

The normalized eigenvectors of the Hadamard operator H have the form

$$|\Psi\rangle_{1,2} = \frac{1}{4\mp 2\sqrt{2}}\left[|1\rangle_1|0\rangle_2 \pm (\sqrt{2}\mp 1)|0\rangle_1|1\rangle_2\right] \quad (11)$$

which agrees with the results for $|\Psi\rangle_1$ and $|\Psi\rangle_2$ in the spin representation.

D. Logical element Y

The logical element Y in the two-boson representation has the form

$$Y = \left(A_1^+A_2^+\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = i\left(A_2^+A_1 - A_1^+A_2\right). \quad (12)$$

Under the action of the oparator Y, the wave function of the qubit |Ψ⟩ transforms to the form

$$i\left(A_2^+A_1 - A_1^+A_2\right)\left(\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2\right) = i\left(\alpha|0\rangle_1|1\rangle_2 - \beta|1\rangle_1|0\rangle_2\right). \quad (13)$$

It can be seen that the coefficient α at the basis vector $|1\rangle_1|0\rangle_2$ goes into $-i\beta$ and the coefficient β at the basis vector $|0\rangle_1|1\rangle_2$ goes into $i\alpha$. The normalized eigenvectors of the operator Ψ have the form

$$|\Psi\rangle_{1,2} = \frac{1}{\sqrt{2}}\left(|1\rangle_1|0\rangle_2 \pm i|0\rangle_1|1\rangle_2\right) \quad (14)$$

E. Logical phase element T

The complex logical phase element T (which is also denoted by π/8) in the spinor representation is given by the matrix

$$T = \begin{pmatrix}1 & 0\\ 0 & \exp(i\pi/8)\end{pmatrix}. \quad (15)$$

In the two-boson representation, the logical element T is determined by the expression

$$T = \left(A_1^+A_2^+\right)\begin{pmatrix}1 & 0\\ 0 & \exp(i\pi/8)\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = A_1^+A_1 + \exp(i\pi/8)A_2^+A_2 \quad (16)$$

The operator T acts on the qubit wave function according to the rule:

$$T|\Psi\rangle = \left[A_1^+A_1 + \exp(i\pi/8)A_2^+A_2\right](\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2) = \alpha|1\rangle_1|0\rangle_2 + \beta\exp(i\pi/8)|0\rangle_1|1\rangle_2. \quad (17)$$

Thus, the operator T does not change the coefficient at the basic vector $|1\rangle_1|0\rangle_2$ and changes the phase of the coefficient at the basis vector $|0\rangle_1|1\rangle_2$. It can be shown that the eigenvectors of the operator T coincide with the basic vectors $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$.

F. *Logical phase element S*

In the two-bosonic representation, the logical phase element S (not to be confused with the spin operator) is defined by the expression

$$S = \left(A_1^+ A_2^+\right)\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = A_1^+A_1 + iA_2^+A_2. \quad (18)$$

Under the action of the operator S, the wave function of the qubit is transformed as follows

$$(A_1^+A_1 + iA_2^+A_2)(\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2) = \alpha|1\rangle_1|0\rangle_2 + i\beta|0\rangle_1|1\rangle_2 \quad (19)$$

Transformation (19) does not change the coefficient α at the basis vector $|1\rangle_1|0\rangle_2$, but multiplies by *i* the coefficient β which is at the basis vector $|0\rangle_1|1\rangle_2$. The eigenvectors of the operator S coincide with the basic eigenvectors $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$.

G. *Logical phase element Φ*

By definition, the logical phase element Φ is given in the two-bosonic representation by the operator

$$\Phi = \left(A_1^+ A_2^+\right)\begin{pmatrix}1 & 0\\ 0 & \exp(i\varphi)\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = A_1^+A_1 + \exp(i\varphi)A_2^+A_2. \quad (20)$$

where φ is the angle of rotation of the state vector of the qubit $|\Psi\rangle$, starting at the center of the Bloch sphere and ending at the surface of the sphere. Rotation is around the z-axis. The basis vector $|1\rangle_1|0\rangle_2$ is oriented along this axis, and the basis vector $|0\rangle_1|1\rangle_2$ is oriented in the opposite direction. The action of the operator Φ on the qubit state vector is determined by the expression

$$\Phi|\Psi\rangle = \left[A_1^+A_1 + \exp(i\varphi)A_2^+A_2\right](\alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2) = \alpha|1\rangle_1|0\rangle_2 + \beta\exp(i\varphi)|0\rangle_1|1\rangle_2. \quad (21)$$

According to (21), the action of the operator Φ on the state vector $|\Psi\rangle$ of the qubit does not change the coefficient α at the basis vector $|1\rangle_1|0\rangle_2$ and multiplies the coefficient β at the basis vector $|0\rangle_1|1\rangle_2$ by the factor $\exp(i\varphi)$. The eigenvectors of the operator Φ coincide with the basis vectors $|1\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$.

IV. EPR STATES IN TWO-BOSON REPRESENTATION OF THE EFFECTIVE SPIN

Let a quantum circuit contains the quantum logic element CNOT (I is the unitary operator):

$$CNOT = \begin{pmatrix} I & 0\\ 0 & A_1^+A_2 + A_2^+A_1\end{pmatrix}, \quad (22)$$

in the control input of which the Hadamard element H is included. If two base vectors $|1\rangle_1|0\rangle_2$ are applied to both inputs of the circuit, then one of them will go immediately to the controlled input of element CNOT, while the second will go to the control input of the CNOT after passing the element H (Fig. 1).

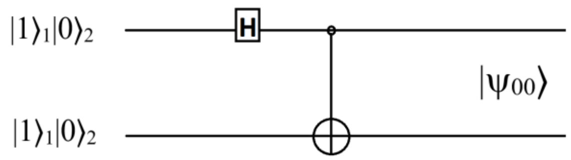

Figure 1. Quantum scheme for creating the EPR [5] or Bell [6] states.

The gate H transforms the basis vector $|1\rangle_1|0\rangle_2$ as follows:

$$H|1\rangle_1|0\rangle_2 = \frac{1}{\sqrt{2}}[A_1^+(A_1 + A_2) + A_2^+(A_1 - A_2)]|1\rangle_1|0\rangle_2 = \frac{1}{\sqrt{2}}\left(|1\rangle_1|0\rangle_2 + |0\rangle_1|1\rangle_2\right) \quad (23)$$

It can be shown that in this case, the input of logical element CNOT is the vector $\frac{1}{\sqrt{2}}\left(|3\rangle_1|1\rangle_2 + |1\rangle_1|2\rangle_2\right)$ which, after passing the element CNOT, is converted to the Bell state

$$|\psi_{00}\rangle = \frac{1}{\sqrt{2}}\left(|3\rangle_1|0\rangle_2 + |0\rangle_1|3\rangle_2\right) \quad (24)$$

The other three Bell states are found similarly in the two-boson representation:

$$\left|\psi_{01}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|2\right\rangle_1\left|1\right\rangle_2 + \left|1\right\rangle_1\left|2\right\rangle_2\right)$$

$$\left|\psi_{10}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|3\right\rangle_1\left|0\right\rangle_2 - \left|0\right\rangle_1\left|3\right\rangle_2\right) \quad (25)$$

$$\left|\psi_{11}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|2\right\rangle_1\left|1\right\rangle_2 - \left|1\right\rangle_1\left|2\right\rangle_2\right)$$

DISCUSSION AND CONCLUSIONS

In the two-boson representation of an effective spin S, a one-to-one correspondence between the discrete energy spectrum of a spin system with a finite number of degrees of freedom and the energy spectrum of two harmonic oscillators with an infinite number of degrees of freedom is possible only in one case.

To do this, it is necessary to introduce a limit on the number of oscillatory states. Namely, to depict spin states using the states of two harmonic oscillators, only those bosonic states must be involved that satisfy the condition $n_1 + n_2 \le 2S$, where n_1 and n_2 are the occupation numbers of the bosonic states related to oscillators 1 and 2. Taking into account (2), the 2S + 1 spin states acquire in the two-boson representation the form
$\left|2S\right\rangle_1\left|0\right\rangle_2, \left|2S-1\right\rangle_1\left|1\right\rangle_2, ..., \left|S+M\right\rangle_1\left|S-M\right\rangle_2, ...,$
$\left|1\right\rangle_1\left|2S-1\right\rangle_2, \left|0\right\rangle_1\left|2S\right\rangle_2.$

Based on the above, the following conclusions can be drawn:

1. The Schwinger's oscillator model of angular momentum can be used to determine logical quantum elements, as well as logical quantum circuits of single- and multi-qubit systems in quantum computing.
2. The application of the two-boson representation of angular momentum to quantum computing may be useful due to the peculiarities of this representation for one- and multi-qubit systems
3. In the case of N-qubit systems, the form of spin operators in the two-boson representation does not depend on the value of N that can lead to simplifications in quantum calculations in some particular cases.

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