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STATISTICAL SIMULATION OF RELIABILITY OF NETWORKS WITH EXPONENTIALLY DISTRIBUTED UNIT LIFETIMES

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Abstract. In this paper, there were deduced three new lifetime distributions of serial-parallel and parallel-serial networks, their distribution being approached by means of analytical and Monte-Carlo methods. The novelty of the distribution consists in the fact that the number of subnets is random, governed by the Poisson and Logarithmic distributions, the lifetimes of the units in each subnet being independent, identically, exponentially distributed random variables, the number of units in each subnet is the same constant integer number. It was shown that the most important theoretical characteristics of lifetime for such networks, as the mean value, the variance, the survival/reliability function, my be approximated, with desired accuracy, by the same corresponding characteristics, as the sample mean value, sample variance, empirical survival/reliability function simulated by Monte-Carlo methods. Results are illustrated tabularly and graphically for some concrete examples.

Keywords: *mean value, variance, survival/reliability function, Monte-Carlo methods.*

Abstract. În această lucrare au fost deduse trei noi distribuții de durată de viață a rețelelor de tip serial-paralel și paralel-serial, distribuția acestora fiind abordată prin metode analitice și metode Monte-Carlo. Noutatea distribuției constă în faptul că numărul de subrețele este aleatoriu, guvernat de distribuțiile Poisson și Logaritmică, duratele de viață ale unităților din fiecare subrețea fiind variabile aleatoare independente, identic, exponențial distribuite, numărul de unități din fiecare subrețea. este același număr întreg constant. S-a arătat că cele mai importante caracteristici teoretice ale duratei de viață pentru astfel de rețele, precum valoarea medie, varianța, funcția de supraviețuire/fiabilitate, pot fi aproximate, cu acuratețea dorită, prin caracteristicile corespunzătoare: valoarea medie a eșantionului, varianța eșantionului și funcția empirică de supraviețuire/fiabilitate, simulate prin metode Monte-Carlo. Rezultatele sunt ilustrate tabelar și grafic pentru unele exemple concrete.

Cuvinte cheie: *valoare medie, varianță, funcție de supraviețuire/fiabilitate, metode Monte-Carlo.*

1. Introduction

Serial-parallel and parallel-serial networks are usually found in many works [1-3] as a subsystem within complex networks, such as Wi-Fi, computers, and communication networks. The overall reliability of these larger networks depends on the reliability of these subnets [4]. Although these network types have been extensively studied, existing research predominantly relies on static probabilistic models [5-6]. These models assume constant probabilities of network units remaining operational over time, with a fixed number of units [5] Architecturally, the structure of serial-parallel and parallel-serial networks is shown and described in the source [7].

However, the ever-changing nature of modern networks requires a more flexible approach. Networks are influenced by various factors, including environmental shifts, user behavior, and hardware wear and tear. All of these elements can affect the reliability of network components. As a result, there is an increasing demand for models that account for the dynamic nature of network reliability, considering fluctuations in operational probabilities and variations in the number of network units. In this context, research on serial-parallel and parallel-serial networks offers significant insights into the reliability of dynamic networks [8]. By examining how these simpler network architectures behave in changing environments, researchers can create more robust models for analyzing and predicting the reliability of complex networks. This empowers network engineers to design more resilient systems capable of withstanding the challenges posed by real-world conditions.

In this paper, we consider the dynamic mathematical models of serial-parallel (type A) networks and parallel-serial (type B) networks [9]. For the study we will take variants (type A or B) in which the network units have exponentially distributed lifetimes being independent, identically distributed random variables (*i.i.d.r.v.*) with the cumulative distribution function (*c.d.f.*) $F(x)$ and the number of units in each subnet being the same and equal to N≥2.

Also, the number of subnets is a random variable M of PSD classes, independent of lifetimes of units.

2. Notions and auxiliary results

As the number of subnets is a variable of PSD classes [10], with Poisson or Logarithmic distribution, we will define according to the source [11] the Power Series Distribution.

Definition 1. We say that *M is a Power Series Distributed random variable with power series function* $A(\omega) = \sum_{m\geq 0} a_m \omega^m$ *and power parameter of the distribution* ω, *shortly* $M \in PSD$, if

$$P(M = m) = \frac{a_m \omega^m}{A(\omega)}, a_m \geq 0, m = 0,1,2, \dots \omega \in (0, \tau),$$

where the power series $\sum_{m\geq 0} a_m \omega^m$ is convergent with radius of convergence a positive number τ. As real networks invariably include at least one subnet, or each subnet contains at least one unit, the distribution of parameters must be 0-truncated. So, as a PSD, 0-truncated $Poisson(\omega)$ and $Log(\omega)$ distributions be represented as in [9] in this way.

Table 1

The representative elements of the PSD class for Poisson and Logarithmic truncated distributions

Distribution	a_m	ω	$A(\omega)$	τ
$Poisson^*(\omega)$, $\omega > 0$	$\begin{cases} \frac{1}{m!}, for\ m = 1,2, \dots, \\ 0, \quad for\ m = 0. \end{cases}$	ω	$e^{\omega} - 1$	$+\infty$
$Log(\omega)$ $0 < \omega < 1$	$\begin{cases} \frac{1}{m}, for\ m = 1,2, \dots, \\ 0, \quad for\ m = 0. \end{cases}$	ω	$-\ln(1-\omega)$	1

Then, according to the paper [9] we have the following:

Proposition 1. *The lifetime cumulative distribution functions (c.d.f.) for networks of type A and B, that we will call distributions of type* $Min(Max) - PSD$ *and* $Max(Min) - PSD$ *can be calculated respectively by the general formulas*

$$\boldsymbol{F_{s-p}(x)} = \left[\mathbf{1} - \frac{B\left(\omega\left(1-\left(F(x)\right)^{N}\right)\right)}{B(\omega)}\right] \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) \quad (1)$$

$$\boldsymbol{F_{p-s}(x)} = \left[\frac{B\left(\omega\left(1-\left(1-F(x)\right)^{N}\right)\right)}{B(\omega)}\right] \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) \quad (2)$$

where: *F(x)* is a *c.d.f.* of lifetime for each unit of subnet, ***N*** is the number of units in each of *M* subnets and $B(\omega)$ is a power series function of *r.v. M.*

We will denote the reliability function, also known as the survival function, of a network by $R(x)$, where $R(x) = 1 - F(x)$. Also, we denote by $R_{s-p}(x)$ the reliability of the serial-parallel network, and by $R_{p-s}(x)$ - the reliability of parallel-serial network. The reliability functions of the respective networks can be calculated by the formulas:

$$\boldsymbol{R_{s-p}(x)} = \left[\frac{B\left(\omega\left(1-\left(F(x)\right)^{N}\right)\right)}{B(\omega)}\right] \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) + \boldsymbol{I}_{(-\infty,\mathbf{0}]}(\boldsymbol{x}) \quad (3)$$

$$\boldsymbol{R_{p-s}(x)} = \left[\mathbf{1} - \frac{B\left(\omega\left(1-\left(1-F(x)\right)^{N}\right)\right)}{B(\omega)}\right] \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) + \boldsymbol{I}_{(-\infty,\mathbf{0}]}(\boldsymbol{x}) \quad (4)$$

3. Exponential *Min(Max)-Poisson and Max(Min)-Poisson* mixed distributions as a lifetime distributions

Thus, let consider that $F(x) = \left(1 - e^{-\lambda x}\right) I_{[0,+\infty)}(x), x > 0$. If $M \sim Poisson^{*}(\omega)$, then $P(M = m) = \frac{\omega^{m}}{m!} / (e^{\omega} - 1)$, $B(\omega) = (e^{\omega} - 1)$. So, for (1), by knowing the formula of the function $B(\omega)$, we have that:

$$\boldsymbol{F_{s-p}(x)} = \left[\mathbf{1} - \frac{e^{\omega\left(1-\left(1-e^{-\lambda x}\right)^{N}\right)} - 1}{e^{\omega} - 1}\right] \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) \quad (5)$$

Deriving **Error! Reference source not found.** with respect to *x*, we obtain that the probability density function (*p.d.f.*) $f_{s-p}(x)$ given by the following formula:

$$\boldsymbol{f_{s-p}(x)} = \frac{N\omega\lambda\left(1-e^{-\lambda x}\right)^{N-1} e^{\omega\left(1-\left(1-e^{-\lambda x}\right)^{N}\right)-\lambda x}}{e^{\omega} - 1} \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) \quad (6)$$

In the same way we find that lifetime *c.d.f.* for networks of parallel-serial type:

$$\boldsymbol{F_{p-s}(x)} = \left[\frac{e^{\omega\left(1-e^{-\lambda N x}\right)} - 1}{e^{\omega} - 1}\right] \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) \quad (7)$$

and its *p.d.f.* density function:

$$\boldsymbol{f_{p-s}(x)} = \frac{N\omega\lambda e^{\omega\left(1-e^{-\lambda N x}\right)-\lambda N x}}{e^{\omega} - 1} \boldsymbol{I}_{[\mathbf{0},+\infty)}(\boldsymbol{x}) \quad (8)$$

Due to the fact that lifetime of the Serial-Parallel distribution is a *r.v.*

$$U_{s-p} = \min[\max(X_{11}, X_{12}, \dots X_{1N}), \max(X_{21}, X_{12}, \dots X_{N2}), \dots, \max(X_{M1}, X_{12}, \dots X_{MN})]$$

and lifetime of the Parallel-Serial distribution is a r.v.

$$V_{p\text{-}s} = max[min(X_{11}, X_{12}, \ldots X_{1N}), min(X_{21}, X_{12}, \ldots X_{N2}), \ldots, min(X_{M1}, X_{12}, \ldots X_{MN})],$$

where: lifetimes of *i-th* unit in the *j-th* subnet X_{ij} are i.i.d.r.v, $X_{ij} \sim exp\{\lambda\}, \lambda>0,$ for $i=1,2,\ldots,N$, N is the number of units in each subnet, and the number of all subnets is a *r.v.* $M \sim Poisson(\omega)$, $\omega > 0$ we say that their corresponding distributions are called, respectively, *Exponential Min(Max)-Poisson and Max(Min)-Poisson mixed distributions*.

Another important indicator is the hazard function [12], also known as failure rate function, that is denoted, for example in the case of Network of type *A*, by $h_{s-p}(x)$ and given by the formula $h_{s-p}(x) = \boldsymbol{f_{s-p}(x)}\left(1 - \boldsymbol{F_{s-p}(x)}\right)$. Applying the last formula to the cases we study, we get

$$\boldsymbol{h_{s-p}(x)} = \frac{N\omega\lambda\left(1-e^{-\lambda x}\right)^{N-1} e^{\omega\left(1-\left(1-e^{-\lambda x}\right)^{N}\right)-\lambda x}}{e^{\omega\left(1-\left(1-e^{-\lambda x}\right)^{N}\right)}-1} \tag{9}$$

and

$$\boldsymbol{h_{p-s}(x)} = \frac{N\omega\lambda e^{\omega\left(1-e^{-\lambda N x}\right)-\lambda N x}}{e^{\omega}\left(1-e^{1-e^{-\lambda N x}}\right)} \tag{10}$$

Remark 1. The lifetime distribution for parallel-serial networks will be excluded from our research because this distribution coincides with the distribution proposed and studied in the paper [13], as a lifetime distribution Exponential Max-Poisson mixed distribution.

Remark 2. Another distribution functions and pdf of the *r.v.* $U_{s\text{-}p}$ and $V_{p\text{-}s}$ for different combination are presented in the paper [3].

So, in the following, we will analyze from a statistical point of view, including through Monte Carlo validation, the serial-parallel type model, thus bringing a new lifetime distribution of serial- parallel networks. Thus, using general formula for lifetime *p.d.f.* **(**6) we will calculate, for our needs, the theoretical mean value and variance numerically, by means of System Mathematica 14.0, because our distribution depend, in fact, of 3 parameters: *λ, ω* and *N*. After we get the results, we will simulate in Mathematica these random variables using Monte-Carlo methods and check how well they approximate the theoretical mean value and the theoretical dispersion [14].

The Monte Carlo simulation algorithm in our case may be described as following:

Step 1: Generate a sample of N values based on exponential distribution with parameter λ.

Step 2: Take the maximum value of sample generated at the Step 1.

Step 3: Generate the value of M using a zero-truncated Poisson distribution with parameter ω.

Step 4: Generate a sample of M values by repeating M times the Steps 1-2.

Step 5: Take the minimum value of the sample created at the Step 4.

Step 6: Calculate, according to the Central Limit Theorem for independent and identically distributed random variables [15], the value $k=[(\sigma x_{1-\alpha/2}/\varepsilon)^2]+1$, where σ is the standard deviation of lifetime $U_{s\text{-}p}$, $x_{1-\alpha/2}$ is the *1-α/2* quantile for standard normal distribution *N(0,1)* and ε is the desired error of approximation for theoretical mean value $\mathbb{E}U_{s\text{-}p}$, taking α=0.05 and ε=0.01.

Step 7: Generate a sample of k values by repeating steps 4-5 k times.

Step 8: Calculate the mean value $\widehat{\mathbb{E}U_{s-p}}$ of the sample generated at the 7th step to approximate the theoretical mean value $\mathbb{E}U_{s\text{-}p}$.

Step 9: Calculate the variance $\mathbb{D}\widehat{U_{s-p}}$ of sample generated at the 7th step to approximate the value of $\mathbb{D}U_{s\text{-}p}$.

Table 2

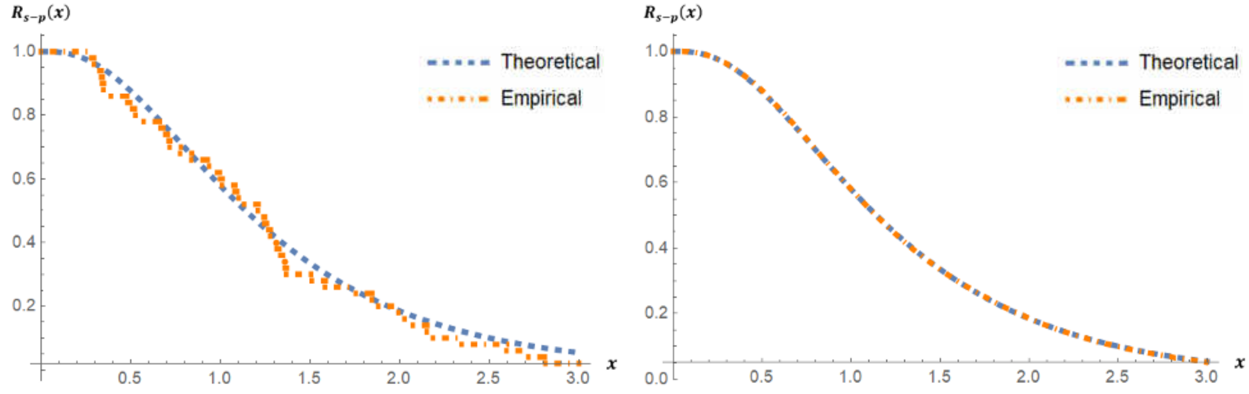
Simulated vs theoretical values of mean and dispersion for Serial-Parallel model with $\boldsymbol{M \sim Poisson^*(\omega)}, \boldsymbol{i.e.} P(M=m) = \frac{1}{e^{\omega}-1}\frac{\omega^m}{m!}, m = 1,2,\dots,\omega > 0$

ω	λ	N	$\mathbb{E}\boldsymbol{U_{s-p}}$	$\mathbb{E}\widehat{\boldsymbol{U_{s-p}}}$	$\mathbb{D}\boldsymbol{U_{s-p}}$	$\mathbb{D}\widehat{\boldsymbol{U_{s-p}}}$
0.5	1.25	3	1.348149323	1.338431878	0.888013261	0.872941761
0.5	1.25	5	1.701888703	1.703793221	0.925795556	0.934127344
0.5	1.25	8	2.04654821	2.044372993	0.946212854	0.942520703
0.5	2.15	3	0.783777285	0.788386991	0.51509177	0.520160698
0.5	2.15	5	0.98782637	0.985834986	0.536698953	0.535302996
0.5	2.15	8	1.187243735	1.195373996	0.547304826	0.557221324
0.5	3.65	3	0.46033396	0.460613746	0.302799361	0.302450773
0.5	3.65	5	0.582155058	0.581128178	0.315834387	0.315438699
0.5	3.65	8	0.700233886	0.699444238	0.324542748	0.320289999
2	1.25	3	1.034266589	1.037160726	0.724065222	0.72071219
2	1.25	5	1.368218002	1.369974019	0.760181813	0.761771466
2	1.25	8	1.702108973	1.69811621	0.78666366	0.789075721
2	2.15	3	0.601482708	0.600396316	0.419109425	0.415211898
2	2.15	5	0.796446843	0.794904434	0.442090831	0.43995408
2	2.15	8	0.991439749	0.989627481	0.460112171	0.460228158
2	3.65	3	0.354989223	0.355135287	0.247767369	0.247998247
2	3.65	5	0.468843135	0.46996092	0.26090492	0.262020569
2	3.65	8	0.582949722	0.584418392	0.269372425	0.268073442
10	1.25	3	0.459026258	0.459073661	0.234353946	0.234169648
10	1.25	5	0.736948531	0.73813245	0.279303413	0.27924198

Note: N – the number of units in each of M subnets; $\mathbb{E}\boldsymbol{U_{s-p}}$ - the theoretical mean value; $\mathbb{E}\widehat{\boldsymbol{U_{s-p}}}$ - the mean value calculated with Monte-Carlo simulated values; $\mathbb{D}\boldsymbol{U_{s-p}}$ - the theoretical variance; $\mathbb{D}\widehat{\boldsymbol{U_{s-p}}}$ - the variance calculated with Monte-Carlo simulated values.

In the Table 2 is shown how for lifetime *p.d.f.* (6) we calculated the theoretical mean value $\mathbb{E}U_{s-p}$ and variance $\mathbb{D}U_{s-p}$ numerically, using the Wolfram Mathematica 14.0 soft, depending on the values of the parameters *λ, ω* and *N*. After that, we simulated them in Wolfram Mathematica using Monte-Carlo methods described in the algorithm above, and obtained for the same parameters, that the empirical mean $\mathbb{E}\widehat{U_{s-p}}$ and empirical variance $\mathbb{D}\widehat{U_{s-p}}$ approximate the theoretical mean value and the theoretical dispersion very accurate, respecting the proposed admissible error ε=0.01. Also, we will follow the similarity in the graphic representation that follows.

It is easy to see how the number of simulations influences the resulting data by viewing the constructed empirical functions. For this we chose two values for the number of simulations k, the first case $k = 50$, and the second case $k = 20735$. The difference between the two cases is that for the second case the number k is calculated according to the Central Limit Theorem [15], as stipulated in the algorithm. In the Figure 2b, we notice that for the same parameters, when the number k was deduced according to the formula the empirical distribution function approximates the theoretical cumulative distribution very well, and for the larger number of iterations (simulated values) they tend to coincide.



a) b)

Figure 2. Survival function for $\omega = 0.5, \lambda = 1.25, N = 3, \varepsilon = 0.01$.

4. Exponential *Min(Max)-Logarithmic* and *Exponential Max(Min)- Logarithmic* mixed distributions as a lifetime distributions

With a similar approach as for the previous case, the study is extended with a new case where the value of M, the number of subnetworks, will be generated using the logarithmic distribution. With the appropriate substitutions, the following results $F(x) = (1 - e^{-\lambda x}) I_{[0,+\infty)}(x), x > 0$. If $M \sim Log(\omega)$, and $P(M = m) = \frac{-1}{\ln(1-\omega)} \cdot \frac{\omega^m}{m}, 0 < \omega < 1, B(\omega) = -\ln(1 - \omega)$ then, for **Error! Reference source not found.**) by knowing the formula of the function $B(\omega)$ and substituting it, we get that:

$$\boldsymbol{F_{s-p}(x)} = \left[\mathbf{1} - \frac{\ln\left(\omega\left((1-e^{-\lambda x})^N - 1\right)+1\right)}{\ln(1-\omega)}\right] \boldsymbol{I_{[0,+\infty)}(x)} \quad (11)$$

The first derivative of $F(x)$ from **Error! Reference source not found.** yields the distribution density function $f(x)$ and it is represented by the formula:

$$\boldsymbol{f_{s-p}(x)} = \frac{N\lambda\omega(1-e^{-x\lambda})^{N-1}}{\left(\omega\left(1-(1-e^{-x\lambda})^N\right)-1\right)\ln(1-\omega)e^{x\lambda}} \quad (12)$$

In the same way, we calculate the lifetime distribution function $F(x)$ for networks of parallel-serial type:

$$\boldsymbol{F_{p-s}(x)} = \frac{\ln\left(1+\left(-1+(e^{-x\lambda})^N\right)\omega\right)}{\ln(1-\omega)} \quad (13)$$

The distribution's density function can be represented as following:

$$\boldsymbol{f_{p-s}(x)} = \frac{\lambda\omega N(e^{-x\lambda})^N}{\left(\left(1-(e^{-x\lambda})^N\right)\omega-1\right)\ln(1-\omega)} \quad (14)$$

Following a similar approach as in the initial case, we utilize the same formula to deduce the hazard function, also known as the failure rate function:

$$\boldsymbol{h_{s-p}(x)} = -\frac{\lambda\omega N(1-e^{-x\lambda})^{N-1}}{e^{x\lambda}\left(\omega(1-e^{-x\lambda})^N-\omega+1\right)\ln\left(\omega(1-e^{-x\lambda})^N-\omega+1\right)} \quad (15)$$

and

$$\boldsymbol{h_{p-s}(x)} = -\frac{\lambda\omega N e^{-\lambda N x}}{(\omega e^{-\lambda N x}-\omega+1)(\ln(1-\omega) - \ln(\omega e^{-\lambda N x}-\omega+1))} \quad (16)$$

In similar fashion, we analyze the next case. With respect to all the given constraints and by an analogous to the first part procedure, we will perform an analysis from a statistical point of view, including validation through Monte Carlo simulation, of the serial-parallel type model and the parallel-serial type model. The main goal is to emphasize the two new lifetime distribution of serial- parallel and parallel-serial networks.

Table 3

Simulated vs theoretical values of mean and dispersion for serial-parallel model with $M \sim Log(\omega)$, i.e., $P(M=m) = \frac{1}{-\ln(1-\omega)} \cdot \frac{\omega^m}{m}, m = 1,2,\ldots, 0<\omega<1$

ω	λ	N	$\mathbb{E}\,U_{s-p}$	$\widehat{\mathbb{E}\,U_{s-p}}$	$\mathbb{D}\,U_{s-p}$	$\widehat{\mathbb{D}\,U_{s-p}}$
0.15	1.25	3	1.428972	1.420671	0.919236	0.916434
0.15	1.25	5	1.784603	1.790588	0.957371	0.962266
0.15	1.25	8	2.132027	2.136382	0.976682	0.978806
0.15	2.15	3	0.829977	0.837779	0.534452	0.53958
0.15	2.15	5	1.03809	1.037251	0.555081	0.55459
0.15	2.15	8	1.238298	1.232948	0.567016	0.568323
0.15	3.65	3	0.48913	0.488425	0.314865	0.311441
0.15	3.65	5	0.611045	0.604446	0.326281	0.319532
0.15	3.65	8	0.729456	0.730239	0.333657	0.336813
0.45	1.25	3	1.324674	1.320241	0.882955	0.883411
0.45	1.25	5	1.676801	1.678226	0.919081	0.920216
0.45	1.25	8	2.022778	2.029045	0.942711	0.947784
0.45	2.15	3	0.771506	0.765819	0.514974	0.508486
0.45	2.15	5	0.975403	0.975921	0.534888	0.535843
0.45	2.15	8	1.1747	1.173953	0.549026	0.548143
0.45	3.65	3	0.454441	0.45114	0.303259	0.300944
0.45	3.65	5	0.574409	0.571679	0.315466	0.312011
0.45	3.65	8	0.691618	0.6897	0.322884	0.32016
0.95	1.25	3	0.893229	0.89389	0.741071	0.741699
0.95	1.25	5	1.212151	1.21096	0.796375	0.790388
0.95	1.25	8	1.534569	1.533734	0.8263	0.824371
0.95	2.15	3	0.520795	0.521037	0.431974	0.433642
0.95	2.15	5	0.704601	0.705342	0.459721	0.460441
0.95	2.15	8	0.894015	0.891345	0.482631	0.476985
0.95	3.65	3	0.30735	0.3059	0.25542	0.25435
0.95	3.65	5	0.415546	0.415356	0.272264	0.27161
0.95	3.65	8	0.524733	0.525343	0.281704	0.281689

Note: N – the number of units in each of M subnets; $\mathbb{E}U_{s-p}$ - the theoretical mean value; $\widehat{\mathbb{E}\,U_{s-p}}$ - the mean value calculated with Monte-Carlo simulated values; $\mathbb{D}U_{s-p}$ - the theoretical variance; $\widehat{\mathbb{D}\,U_{s-p}}$ - the variance calculated with Monte-Carlo simulated values.

Table 3 presents the calculation of the theoretical mean and variance for the lifetime probability density function (*p.d.f.*) given in equation (6). We repeated the numerical calculations using Wolfram Mathematica 14.0, based on the parameters λ, ω, and N. Subsequently, we simulated these values using Monte Carlo methods as described in the preceding algorithm. The results confirm one more time that the empirical mean $\widehat{\mathbb{E}\,U_{s-p}}$ and empirical variance $\widehat{\mathbb{D}\,U_{s-p}}$ closely approximate the theoretical values, adhering to the proposed admissible error of ε=0.01. The following graphical representation further demonstrates this similarity.

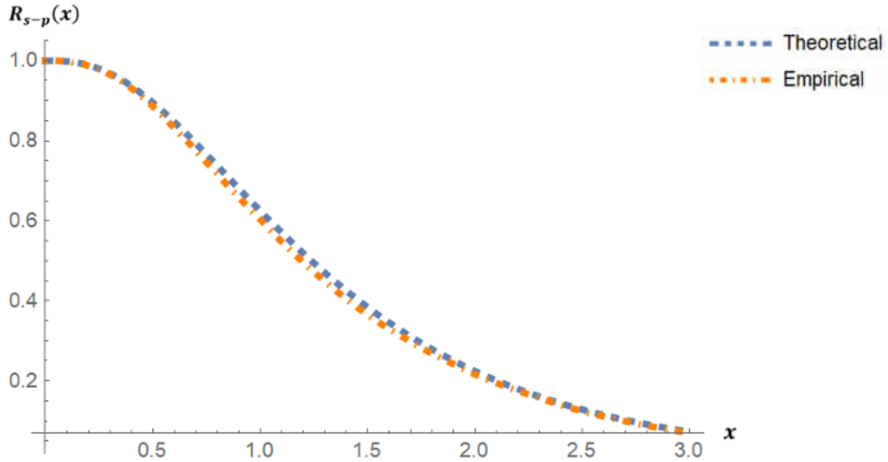


Figure 3. Survival function graph for $\omega = 0.5, \lambda = 1.15, N = 3, \varepsilon = 0.01$.

In the following table of values and graphical example, we observe how well the empirical functions approximates the theoretical values for parallel-serial networks with $M \sim Log(\omega)$.

Table 4

Simulated vs theoretical values of mean and dispersion for serial-parallel model with $M \sim Log(\omega)$, i.e., $P(M=m) = \frac{1}{-\ln(1-\omega)} \cdot \frac{\omega^m}{m}, m = 1,2,\ldots, 0 < \omega < 1$

ω	λ	N	$\mathbb{E}\,V_{p-s}$	$\mathbb{E}\,\widehat{V_{p-s}}$	$\mathbb{D}\,V_{p-s}$	$\mathbb{D}\,\widehat{V_{p-s}}$
0.25	1.25	3	0.28581	0.28576	0.275492	0.277106
0.25	1.25	5	0.171706	0.170882	0.165372	0.164594
0.25	1.25	8	0.107111	0.107962	0.103412	0.102547
0.5	2.15	3	0.183798	0.183041	0.169445	0.168233
0.5	2.15	5	0.110467	0.111056	0.101673	0.101606
0.5	2.15	8	0.069047	0.068979	0.063599	0.063494
0.85	3.15	3	0.166029	0.166184	0.135527	0.135488
0.85	3.15	5	0.0997884	0.100007	0.0813673	0.081666
0.85	3.15	8	0.0623611	0.0623248	0.0508497	0.0506674

Note: N – the number of units in each of M subnets; $\mathbb{E}V_{s-p}$ - the theoretical mean value; $\mathbb{E}\,\widehat{V_{s-p}}$ - the mean value calculated with Monte-Carlo simulated values; $\mathbb{D}V_{s-p}$ - the theoretical variance; $\mathbb{D}\,\widehat{V_{s-p}}$ - the variance calculated with Monte-Carlo simulated values.

For Table 4, we proceeded analogously to the steps for Tables 2 and 3, only we reduced the size of the displayed data set, but we note that the results are just as good, respecting the chosen error. So we continued to compute numerically in Wolfram Mathematica according to the parameters λ, ω, and N the calculation of the theoretical mean and variance, this time according to the lifetime probability density function (*p.d.f.*) given in equation (8) for parallel-serial model. As expected, the theoretical and empirical results tend to coincide. Even better, we see these results interpreted graphically below.

What we see in Figure 4 is that for $k = 34447$ simulations, calculated according to the Central Limit Theorem [15], with the help of Wolfram Mathematica 14.0 software we obtained these estimates of the empirical and theoretical survival function. We notice that for sufficient number of simulations, these two functions tend to coincide.

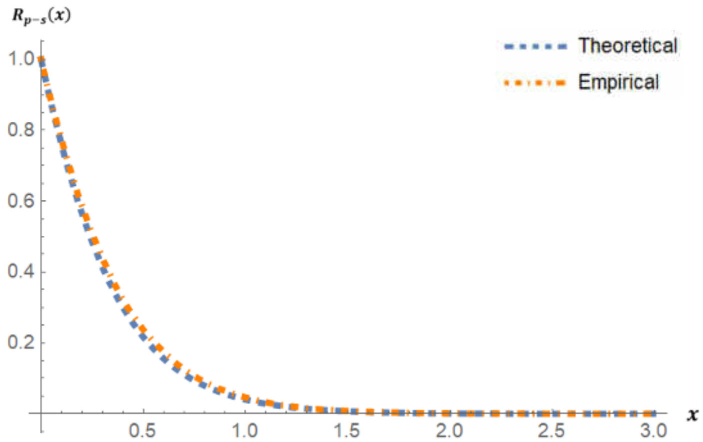


Figure 4. Survival function for $\omega = 0.5, \lambda = 1.25, N = 3, \varepsilon = 0.01$

5. Conclusion

The changing nature and evolution of networks encountered in engineering today lead to an increased demand for dynamic reliability analysis models. These models are complex and different in operation, requiring the models to be tailored to the problem, the network structure and how the subnets and units work. Starting from the goal of analyzing the reliability of networks with serial-parallel and parallel-serial architecture, we managed to derive 3 new lifetime distributions, distributions approached by analytical and empirical methods.

The studied dynamic networks have serial-parallel or parallel-serial architecture, each time the number of subnets being a random variable M of PSD classes, with Poisson or Logarithmic distribution. The number of units in each subnet is constant, greater than or equal to 2, the lifetimes of the units in each subnet being independent, identically, exponentially distributed random variables. In the aforementioned conditions, based on the general formulas for the calculation of the *c.d.f.*, the calculation formulas for the cumulative distribution function, the reliability function and the hazard for these new three lifetime distributions were deduced.

Next, based on the deduced formulas, the theoretical mean value and the theoretical dispersion were calculated numerically, implementing the corresponding functions available in Wolfram Mathematica. Also, with the help of this software we simulated for the same parameters, random variables through the Monte-Carlo method and obtained the empirical mean and the empirical dispersion. We have presented the obtained data in tabular and graphical form, from where we can see that the theoretical and empirical values are very close, the difference between them does not exceed the admissible error shifted by us in the algorithm, and for a fairly large sample, these values tend to coincide. Thus, we validated the obtained theoretical results.

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