CONFORMALLY-FLAT SOLUTIONS FOR A RADIATIVE RELATIVISTIC SPHERE

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INTRODUCTION

The exact solutions of a relativistic fluid play a more important role than those obtained through approximation scheme and numerical approximation. Moreover, one uses various symmetries to get physical viable information from the complicated structure of the field equations in Einstein's theory. Solutions of the Einstein field equations for a perfect fluid with or without a radiation field have been studied by several authors [1]-[9]. Owing to the nonlinearity of the field equations, it is very difficult to obtain exact solutions.

In the present paper, we present a conformally flat metric representing the gravitational field of a spherically symmetric distribution of a radiating perfect fluid. A particular case of the solution is discussed and corresponding expressions for fluid energy density, pressure, radiation flux and radiation energy density have been derived.

The solutions of the equations of the relativistic perfect fluids are analyzed as relativistic models of a radiating balanced sphere. In comoving coordinates in which we choose the units so that $c = 1$, the metric of a conformally flat space-time for spherically symmetric distribution can be written as [8]

$$ds^2 = A^2(t,r)(dt^2 - dr^2 - r^2 d\Omega^2), \qquad (1)$$

where $A = A(t,r)$,

$$d\Omega^2 = d\theta^2 + \sin^2 d\varphi^2,$$

θ and φ labelling points on the unit sphere.

The energy-momentum tensor of a relativistic thermodynamical perfect fluid in the presence of a radiation field [10], [11] is the sum of the fluid energy-momentum tensor and the energy-momentum tensor of the radiation field.

$$T_\alpha^\beta = w u_\alpha u^\beta - p h_\alpha^\beta + Q_\alpha^\beta, \alpha, \beta = 0,3, \qquad (2)$$

with

$$Q_\alpha^\beta = Q u_\alpha u^\beta + q_\alpha u^\beta + u_\alpha q^\beta - \frac{1}{3} Q h_\alpha^\beta, \qquad (3)$$

where $h_\alpha^\beta = g_\alpha^\beta - u_\alpha u^\beta$ is the spatial projector, $u_\alpha = (A,0,0,0)$ is the fluid velocity, Q is the density of the radiation energy and $q_\alpha = (0, qA, 0, 0)$ is the radiative flux.

The Einstein field equations [12]

$$R_\alpha^\beta - \frac{1}{2} R \delta_\alpha^\beta = \kappa T_\alpha^\beta, \qquad (4)$$

(κ being Einstein's gravitational constant) which connect the Ricci tensor R_α^β with the energy-momentum tensor T_α^β given by (2), become:

$$3\left(\frac{A_t}{A}\right)^2 + \left(\frac{A_r}{A}\right)^2 - \frac{2A_{rr}}{A} - \frac{4A_r}{rA} = \kappa A^2 w^*, \qquad (5)$$

$$-\left(\frac{A_t}{A}\right)^2 + \frac{2A_{tt}}{A} - 3\left(\frac{A_r}{A}\right)^2 - \frac{4A_r}{rA} = \kappa A^2 p^*, \qquad (6)$$

$$\frac{A_{tt}}{A} + \left(\frac{A_t}{A}\right)_t - \frac{A_{rr}}{A} - \frac{2A_r}{rA} - \left(\frac{A_r}{A}\right)_r = \kappa A^2 p^*, \qquad (7)$$

$$2\left(\frac{A_{tr}}{A} - 2\frac{A_t}{A}\frac{A_r}{A}\right) = \kappa A^2 p^*, \qquad (8)$$

where $w^* = w + Q$, $p^* = p + \frac{1}{3}Q$ the total energy and the total pressure. The conservation identities $T_{\alpha;\beta}^\beta = 0$ become

$$w_t + 3\frac{A_t}{A}(w + p) = \sigma(Q - aT^4), \qquad (9)$$

$$p_r + \frac{A_r}{A}(p + w) = \sigma A q. \qquad (10)$$

The equations of radiation field

$$Q_{\alpha;\beta}^\beta + F_\alpha = 0,$$

with $F_\alpha = \sigma\left(q_\alpha + (Q - aT^4)\right)u_\alpha$, can be written as

$$Q_t + \frac{A_t}{A}Q + q_r + 2\left(2\frac{A_r}{A} + \frac{1}{r}\right)q = -\sigma A(Q - aT^4), \qquad (11)$$

$$q_t + 4\frac{A_t}{A}q + \frac{1}{3}\left(Q_r + 4\frac{A_r}{A}Q\right) = -\sigma A q. \qquad (12)$$

The equations (9)-(12) give

$$w_t^* + 3\frac{A_t}{A}\left(w^* + p^*\right) + q_r + 2\left(2\frac{A_r}{A} + \frac{1}{r}\right)q = 0, \qquad (13)$$

$$p_t^* + \frac{A_r}{A}\left(p^* + w^*\right) + q_t + \frac{A_t}{A}q = 0, \qquad (14)$$

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The elimination of p^* from (6) and (7) gives

$$\frac{A_{rr}}{A} - 2\left(\frac{A_r}{A}\right)^2 - \frac{1}{3}\frac{A_r}{A} = 0.$$

From here and (8) we obtain the system

$$\left(\frac{A_r}{rA^2}\right)_r = 0, \quad \left(\frac{A_r}{rA^2}\right)_t = \frac{\kappa}{2}\frac{A}{2}q. \qquad (15)$$

The compatibility condition is

$$\left(\frac{\kappa}{2}\frac{A}{2}q\right)_r = 0,$$

so that $\frac{\kappa}{2}\frac{A}{2}q = -2F'(t),$ then

$$\frac{A_r}{A^2} = -2rF(t),$$

whose solution is

$$A(t,r) = \left(F(t)r^2 + G(t)\right)^{-1}, \qquad (16)$$

where $F(t)$ and $G(t)$ are arbitrary functions.

A relation between the functions $F(t)$ and $G(t)$ can be obtained from the condition that on the hypersurface $r = r_s,$ the total pressure $p^*(t,r_s) = 0.$ In this condition, from the equation (6), we obtain

$$-2\left[F(t)r_s^2 + G(t)\right]\left[F''(t)r_s^2 + G''(t)\right] + 3\left[F'(t)r_s^2 + G'(t)\right] = 4F(t)\left[F(t)r_s^2 - 2G(t).\right]$$

(17)

Many solutions have been found by specifying a functional relation between some metric functions. We shall present models of radiating sphere by considering the particular case $G(t) = kF(t).$ In this case, from the equation (17), we obtain for F

$$\left(r_s^2 + k\right)^2\left[-2F(t)F''(t) + 3\left(F'(t)\right)^2\right] = 4\left(r_s^2 - 2k\right)F^2(t).$$

This gives

$$A(t,r) = \frac{1}{r^2 + k}\frac{1}{F(t)} = \frac{1}{r^2 + k}\left(C_1e^{\alpha t} + C_2e^{-\alpha t}\right)^2, \qquad (18)$$

with $\alpha^2 = \frac{r_s^2 - 2k}{\left(r_s^2 + k\right)^2},$ where C_1 and C_2 are arbitrary constants.

The fluid energy density, pressure and radiation flux can then be computed from (5)-(8):

$$w^*(t,r) = \frac{12}{\kappa}\frac{\left(r^2 + k\right)^2}{\left(C_1e^{\alpha t} + C_2e^{-\alpha t}\right)^4} \times \left[\alpha^2\left(\frac{C_1e^{\alpha t} - C_2e^{-\alpha t}}{C_1e^{\alpha t} + C_2e^{-\alpha t}}\right)^2 + \frac{k}{\left(r^2 + k\right)^2}\right], \qquad (19)$$

$$p^*(t,r) = \frac{4}{\kappa}\frac{\left(r^2 + k\right)^2}{\left(C_1e^{\alpha t} + C_2e^{-\alpha t}\right)^4} \times \left[\frac{r^2 - 2k}{\left(r^2 + k\right)^2} - \alpha^2\right], \qquad (20)$$

$$q(t,r) == \frac{8\alpha r}{\kappa}\left(r^2 + k\right)\frac{C_1e^{\alpha t} - C_2e^{-\alpha t}}{\left(C_1e^{\alpha t} + C_2e^{-\alpha t}\right)^5}. \qquad (21)$$

If we substitute $A(t,r)$ given by (18) in (11})-(14), we obtain

$$w_t^* + 6\alpha\frac{C_1e^{\alpha t} - C_2e^{-\alpha t}}{C_1e^{\alpha t} + C_2e^{-\alpha t}}\left(w^* + p^*\right) + q_r - 2\frac{3r^2 - k}{\left(r^2 + k\right)}q = 0, \qquad (22)$$

$$p_r^* - \frac{2r}{r^2 + k}\left(w^* + p^*\right) + q_r + 8\alpha\frac{C_1e^{\alpha t} - C_2e^{-\alpha t}}{C_1e^{\alpha t} + C_2e^{-\alpha t}}q = 0, \qquad (23)$$

$$Q_t + 8\alpha\frac{C_1e^{\alpha t} - C_2e^{-\alpha t}}{C_1e^{\alpha t} + C_2e^{-\alpha t}}Q + q_r - -2\frac{3r^2 - k}{r\left(r^2 + k\right)}q = -\alpha\frac{1}{r^2}\left(C_1e^{\alpha t} + C_2e^{-\alpha t}\right)^2\left(Q - aT^4\right), \qquad (24)$$

$$q_t + 8\alpha\frac{C_1e^{\alpha t} - C_2e^{-\alpha t}}{C_1e^{\alpha t} + C_2e^{-\alpha t}}q + \frac{1}{3}\left(Q_r - \frac{8rQ}{r^2 + k}\right) = -\frac{\sigma q}{r^2}\left(C_1e^{\alpha t} + C_2e^{-\alpha t}\right)^2. \qquad (25)$$

In virtue of the conservation identities, the functions $w^*,$ p^*, and $q,$ given by (19)-(21), verify the equations (22) and (23}). With q given by (21), the equation (25) can be written

$$\left(\frac{Q}{\left(r^2+k\right)^4}\right)_r + +B(t)\frac{r}{\left(r^2+k\right)^3}+C(t)\frac{1}{r\left(r^2+k\right)^3}=0,$$
26)

where

$$B(t)=\frac{24\alpha^2}{\kappa}\times$$
$$\times\frac{\left(C_1e^{\alpha t}+C_2e^{-\alpha t}\right)^2+3\left(C_1e^{\alpha t}-C_2e^{-\alpha t}\right)^2}{\left(C_1e^{\alpha t}+C_2e^{-\alpha t}\right)^6},$$
$$C(t)=\frac{24\sigma\alpha}{\kappa}\frac{C_1e^{\alpha t}-C_2e^{-\alpha t}}{\left(C_1e^{\alpha t}+C_2e^{-\alpha t}\right)^3}.$$

The general solution of the equation (26) is

$$Q(t,r)=\frac{1}{4}B(t)\left(r^2+k\right)^2-C(t)\left(r^2+k\right)^4\times$$
$$\times\left[\frac{1}{2k^3}ln\frac{r^2}{r^2+k}+\frac{1}{4}\frac{2r^2+3k}{\left(r^2+k\right)^2}\right],$$

for $k\neq 0,$ and

$$Q(t,r)=\frac{1}{4}B(t)r^4+\frac{1}{6}C(t)r^2+Q_0(t)r^8,$$

for $k=0.$

The function $Q_0(t)$ can be determined from the condition

$$Q(t,r_s)=Q_s(t).$$

For example, if $k=0,\ C_1=1,\ C_2=0,$ then

$$Q(t,r)=$$
$$=Q_s(t)\left(\frac{r}{r_s}\right)^8+\frac{24}{\kappa}\frac{r^8}{r_s^2}\left(\frac{1}{r^4}-\frac{1}{r_s^4}\right)e^{-\frac{4}{r_s}t}+$$
$$+\frac{4\sigma}{\kappa}\frac{r^8}{r_s}\left(\frac{1}{r^6}-\frac{1}{r_s^6}\right)e^{-\frac{2}{r_s}t}.$$

If we choose in (16) $F(t)$ a constant ($F(t)=1$), one finds

$$A(t,r)=\frac{1}{r^2-(t-b)^2},$$

with b an arbitrary constant, which corresponds to a solution of the Einstein's vacuum equations [6].

The equation (24) gives the relation between the density of radiation energy $Q(t,r)$ and the temperature $T(t,r).$

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