# Non-Euclidean Geometry of Human Body and Electrical Networks

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***Abstract* — The review of application of non-Euclidean geometries for interpretation of growth of the human body is submitted and features of use of non-Euclidean geometries in the electric circuit theory are shown. The common mathematical apparatus represents interdisciplinary approach in view of analogy of processes of a different physical nature. Growth of the human body and changes of parameters of an operating regime of a network correspond to projective and conformal transformations which possess an invariant, that is the cross-ratio of four points. The obtained results develop methodology of application of non-Euclidean geometries.**

*Index Terms* —**human body, cross- ratio, Möbius equivalent, network regime, non- Euclidean geometry.**

## I. INTRODUCTION

The growth of the human body is essentially nonlinear for ordinary or Euclidean geometry. Analysis of the structure and growth reveals that separate parts or all the three-component kinematics blocks (the phalanxes of fingers, the three-membered extremities and the three-membered body) change according to Möbius transformations which are characteristic for conformal and projective geometry.

Such transformations possess an invariant or invariable value. Therefore, all the called three-component blocks are characterized by a constant value throughout life [1], [2].

From this point of view, Euclidean geometry appears as one of the possible geometry. The advent of the special theory of relativity leads to a new term, "geometrization of physics", which stood for the fact that formally this theory is the theory of invariants of some group of transformations (Poincaré-Lorentz group), or geometry.

The basic principles of projective geometry are reflected in methods of linear perspective. As the result of its application, there is an image of a subject in which both distances and corners change. But these changes happen not arbitrarily; the cross- ratio of four points on a straight line (a double proportion) remains invariable.

Geometry of the Euclidean space complemented by one infinitely remote point is called as conformal geometry [3]. Projective interpretation in the form of a stereographic projection of points of the sphere to the plane gives an example of the conformal plane (an easily understood example is cartography) in which the value of the cross- ratio of four points remains [4]. This plane is applied in physics for solving of electrostatic problem, that is the method of mirror images.

Möbius's certain subgroup of transformations corresponds to Lobachevskian geometry or hyperbolic geometry, that is Poincare's so-called conformal interpretation [5].

It appears that the space of visual perception is characterized by Lobachevskian geometry.

In a number of papers it is shown that in electric networks the changes of operating regime parameters it is possible to interpret as projective and conformal transformations.

In addition, relationship of regime parameters at different parts of a network also is being described by projective transformations [6], [7], [8].

It is natural, when the common mathematical apparatus is applied in various areas of science; it represents interdisciplinary approach in view of analogy of processes of a different physical nature.

As the example of this interdisciplinary approach, the review of application of non-Euclidean geometries for interpretation of growth of the human body is submitted and features of use of non-Euclidean geometries in the electric circuit theory are shown.

## II. SOME FACTS ABOUT USED GEOMETRIC TRANSFORMATIONS

***Projective transformations. General case.*** Generally, the projective transformation of points of one straight line $U_L$ into points of other line $R_L$ is set by the projection centre $S$ or the three pairs of respective points in Fig.1. The projective transformations preserve the cross ratio of four points

$$m = (R_L^1\ R_L^2\ R_L^3\ R_L^4) = \frac{R_L^2 - R_L^1}{R_L^2 - R_L^4} \div \frac{R_L^3 - R_L^1}{R_L^3 - R_L^4},$$

$$m = (U_L^1\ U_L^2\ U_L^3\ U_L^4).$$

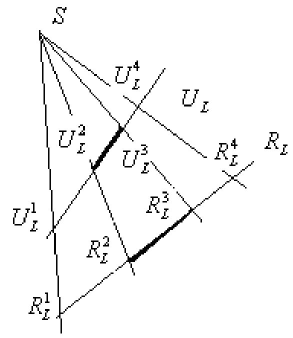

Fig.1. Projective transformation of straight lines points

***Affine transformation.*** There is the projection centre $S$, but the straight lines $U_L$, $R_L$ are parallel. Therefore, the invariant of an affine transformation is the simple ratio or proportion of three points.

***Euclidean transformation***. If the projection centre $S \to \infty$, and the straight lines $U_L$, $R_L$ are parallel, the projection is carried out by parallel lines. This projection corresponds to the Euclidean transformation, that is parallel translation of a segment. The Euclidean transformation preserves the difference of points.

The considered transformations of an initial rectangular coordinate grid are presented in Fig.2.

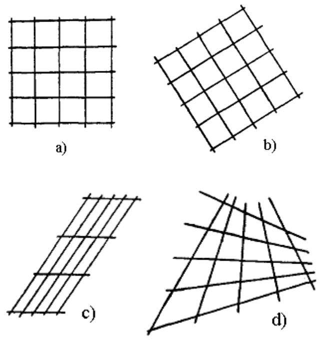


Fig.2. Characteristic transformation of the Cartesian grid -a) by various group transformations: Euclidean -b) , affine -c), projective -d)

***Stereographic projection***. The projection of points of the sphere $U_i(U_1, U_2)$ from the top pole on the tangent plane $n_1$, $n_2$ at the bottom pole is presented in Fig.3. For simplicity of figure, the coordinate axes $U_1, U_2$ and $n_1$, $n_2$ are combined.

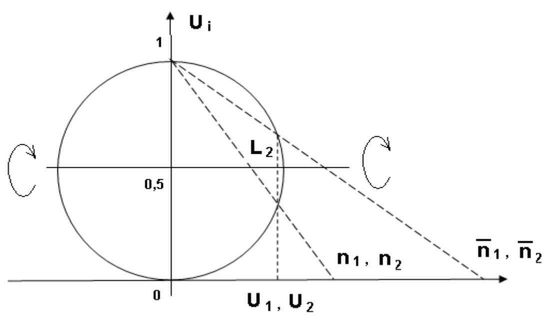


Fig.3. Stereographic projection of the sphere on the conformal plane $n_1$, $n_2$

The conformal plane differs from Euclidean by existence of one infinitely remote point corresponding to the top pole of sphere.

***Conformal transformations.*** The area of change of values $U_1, U_2$ corresponds to the sphere equator in Fig.4,a.

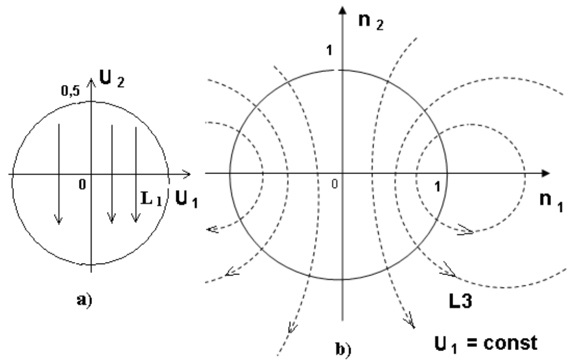


Fig.4. Correspondence the plane $U_1, U_2$ – a) and conformal plane $n_1$, $n_2$ - b) for $U_1 = const$

Let the value $U_1$ be $U_1 = const$ (that is the line $L_1$). Then, the circular section $L_2$ on the sphere and the circle $L_3$ on the plane $n_1$, $n_2$ turn out. The similar family of circles is described by the rotation group of sphere, as it is shown by arrows. By definition, Möbius's group of transformations preserve the values of angles and transform spheres into spheres. In addition, Möbius's transformations are locally similar transformations, Fig.5.

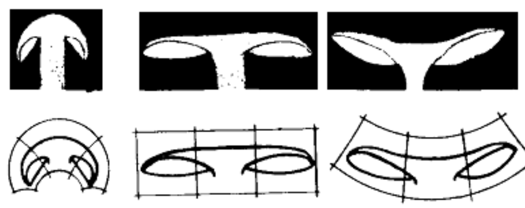

Fig.5. Growth transformations in cap of fungus and their modeling as Möbius's transformations

## III. GROWTH CHANGES OF VALUES OF THE HUMAN BODY AS NON- EUCLIDEAN TRANSFORMATIONS

Nonlinear transformations of human skull with aging are described as Möbius's transformations in Fig.6.

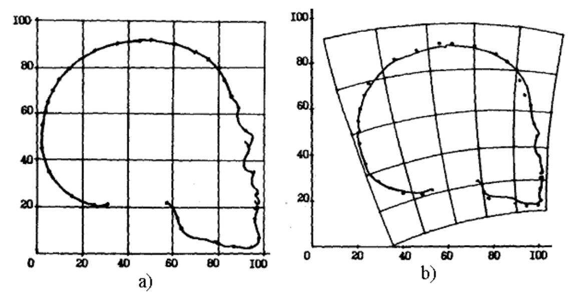


Fig.6. Möbius's transformations in the modeling of ontogenetic transformations of the human skull. Profiles of the skull of an adult –a) and 5-year old –b)

Growth changes of three-component kinematic blocks (the phalanxes of fingers, the three- membered extremities and the three- membered body) are characterized by the constant value of cross-ratio of four points. In particular, for the fingers the cross- ratio has the view

$$W = \frac{(C-A)\cdot(D-B)}{(C-B)\cdot(D-A)} = const,$$

The expressions in parentheses are the lengths of segments between the end points of finger phalanxes. $AB, BC, CD$ - are the lengths of basic phalanx, middle and end phalanxes.

The values of cross-ratio of all the blocks, at least during individual development, are grouping around the benchmark 1.31. Meanwhile, the growth of the human body is essentially nonlinear, Fig.7.

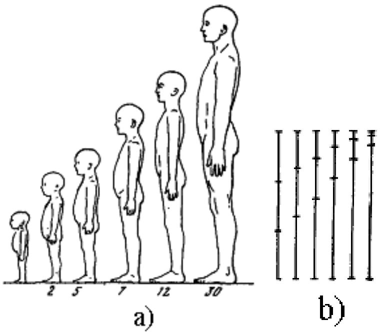


Fig.7. Changes of the human body with age - a), the three-stretch parts cross- ratios are equal to1,31 -б)

Thus, all the three-membered blocks of the human kinematics are Möbius equivalent and Möbius invariable during the lifetime.

## IV. PROJECTIVE TRANSFORMATIONS OF ELECTIC NETWORK. KINDS OF CROSS- RATIOS

Let an electric circuit (an active two-pole $A$) in Fig.8 be considered. At change of load $R_L > 0$ from a regime of short circuit *SC* $(R_L = 0)$ to the open circuit *OC* $(R_L = \infty)$, the load straight line or *I-V* characteristic $I_L(U_L)$ is obtained. Further, it is possible to calibrate the *I-V* characteristic by load resistance values.

The equation $V_L(R_L)$ has the characteristic linear-fractional view

$$V_L = V_0 \frac{R_L}{R_i + R_L}, \qquad (1)$$

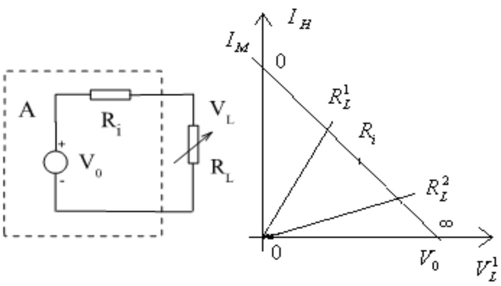


Fig.8. Electric circuit with variable load and its *I-V* characteristic

It gives the grounds for considering the transformation of the straight line $R_L$ into line $V_L$ as projective in Fig.9.

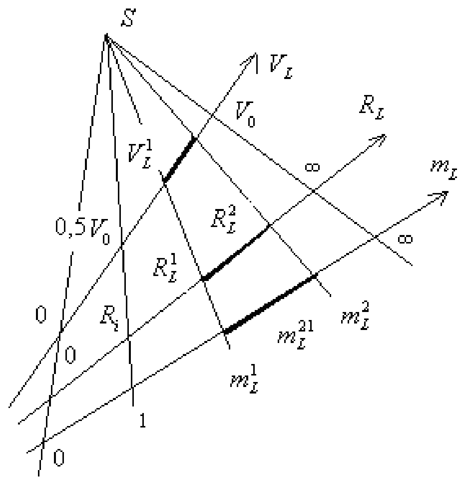


Fig.9. Projective transformation $R_L \rightarrow U_L$ and the cross ratio $m_L$

It is convenient to use the points of characteristic regimes, as the pairs of respective points, that is short circuit, open circuit, maximum load power. If the fourth point is the point of running regime, $R_L^1, V_L^1, I_L^1$, then the cross ratio $m_L$ has the form

$$m_L^1 = (0\ R_L^1\ R_i\ \infty) = \frac{R_L^1 - 0}{R_L^1 - \infty} : \frac{R_i - 0}{R_i - \infty} = \frac{R_L^1}{R_i},$$

$$m_L^1 = (\ 0\ V_L^1\ 0{,}5V_0\ \ V_0\ ) = \frac{V_L^1}{V_0 - V_L^1}. \qquad (2)$$

Thus***, the coordinate of running regime point is set by this value*** $m_L$***, which is defined in the invariant manner through the various regime parameters***, $R_L, V_L$ .

The regime change $R_L^1 \rightarrow R_L^2$ can be expressed similarly

$$m_L^{21} = (0\ R_L^2\ R_L^1\ \infty) = \frac{R_L^2}{R_L^1} = \frac{V_L^2}{V_0 - V_L^2} : \frac{V_L^1}{V_0 - V_L^1}. \qquad (3)$$

Now, let it be necessary to set the identical changes of regime for different initial regimes on the line $V_L$ in Fig. 10.

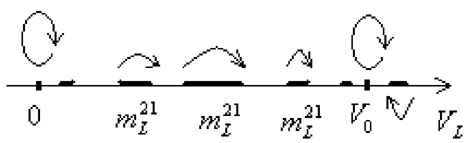


Fig.10. Identical changes of regime for different initial regimes

For this purpose, from (2) we receive the expression $V_L^2(V_L^1)$ explicitly, eliminating $R_i$ for two values $R_L^2, R_L^1$

$$\frac{V_L^2}{V_0} = \frac{m_L^{21} \cdot \frac{V_L^1}{V_0}}{(m_L^{21}-1) \cdot \frac{V_L^1}{V_0} + 1}. \quad (4)$$

The obtained transformation with the parameter $m_L^{21}$ translates the point of initial regime $V_L^1$ into the point $V_L^2$. Therefore, by keeping the parameter of this transformation invariable and by setting different values of initial regime $V_{L1}^1, V_{L2}^1$, etc., we obtain the points of the subsequent regimes $V_{L1}^2, V_{L2}^2$, etc., which form a segment of invariable length (in sense of projective geometry), which is considered as a segment movement in geometry. Here, the character of a change of Euclidean (usual) length of the segment is visible. Approaching to the base points, Euclidean length is decreasing to zero and then is increasing again at the moment of transition to external area. Thus, regime changes are projectively similar for different initial regimes. The received movement of a segment with invariable value of the cross- ratio is similar to Fig.7,b).

In the theory of the projective transformations, an important role is played by the fixed points, which can be accepted as the base points. For their finding, the equation (4) is solved for condition $V_L^1 = V_L^2$. It turns out the two real roots, $V_L = 0,\ V_L = V_0$ which define a hyperbolic transformation and hyperbolic (Lobachevski) geometry, respectively. If roots of the equation coincide, one fixed point defines a parabolic transformation and, respectively, parabolic (Euclidean) geometry. If roots are imaginary, geometry is elliptic (Riemannian).

***Input- output projective conformity of network.*** Let us consider a two-port *TP* in Fig.11. Its equation has the form

$$\begin{pmatrix} V_0 \\ I_0 \cdot \rho \end{pmatrix} = \begin{pmatrix} ch\gamma & sh\gamma \\ sh\gamma & ch\gamma \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ I_1 \cdot \rho \end{pmatrix},$$

where $\gamma$ is an attenuation coefficient, $\rho$ is characteristic or wave resistance. This transformation can be seen as a rotation of the radius-vector $0Y_L$ of constant length at the angle $\gamma$ to the position $0Y_{IN}$ in the pseudo-Euclidean space $I, V$ in Fig.11,b.

Then, we have also the following invariant

$$V_1^2 - I_1^2\rho^2 = V_0^2 - I_0^2\rho^2$$

as length of the vector $0Y_L$. This approach corresponds to Lorenz transformations in mechanics of the relative motion.

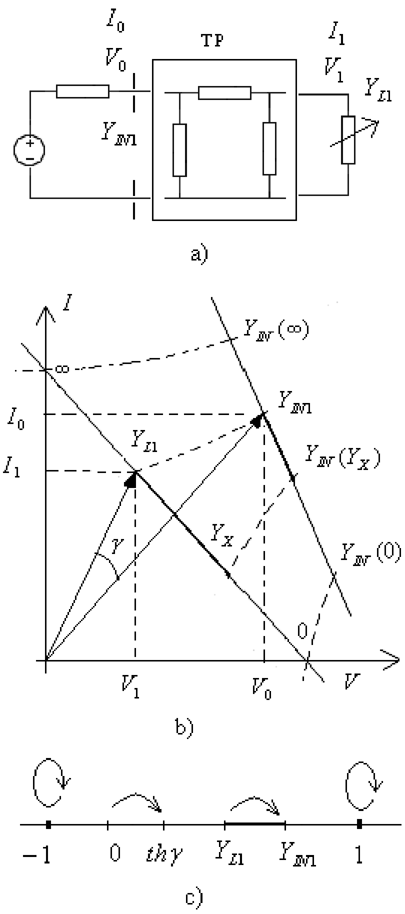


Fig.11. Two-port network - a), input- output characteristic – b), movement of the segment $Y_{L1}\,Y_{IN1}$ for different initial values $Y_{L1}$ -c)

In turn, the conductivities at the input and output are connected already by linear- fractional expression

$$Y_{IN1}\rho = \frac{Y_{L1}\rho + th\gamma}{1 + Y_{L1}\rho \cdot th\gamma}.$$

This expression corresponds to the rule of addition of relativistic velocities. Let us consider the points $Y_{L1}, Y_{IN1}$ on superposed axis in Fig.11,c. This figure represents a point movement from the position $Y_{L1}$ to the position $Y_{IN1}$ (or the segment movement $Y_{L1}\,Y_{IN1}$) for different initial values $Y_{L1}$ as shown by arrows. Then, the points $\pm 1$ are fixed. The cross-ratio of the points $Y_{L1}, Y_{IN1}$, relatively fixed points, determines the "length" of segment $Y_{L1}\,Y_{IN1}$ or the maximum of efficiency $K_{PM}$ of a two-port

$$m\ (Y_{L1}\,Y_{IN1}) = \frac{1 - th\gamma}{1 + th\gamma} = K_{PM}.$$

Thus, one more received invariant equal to concrete number. Therefore, this invariant is similar to the cross-ratio of the human body equal to 1.31.

***Cascade connection of two-ports.*** Let us consider the cascaded two-ports *TP1* and *TP2* in Fig.12. The relationship of regime parameters at different parts of the network or "movement" on these parts also corresponds to projective transformations. The load change from the value $Y_{L2}^1$ to the value $Y_{L2}^2$ defines the correspond changes $Y_{L1}, Y_{IN1}$. The length of segments of all the load lines is different for the usually used Euclidean geometry.

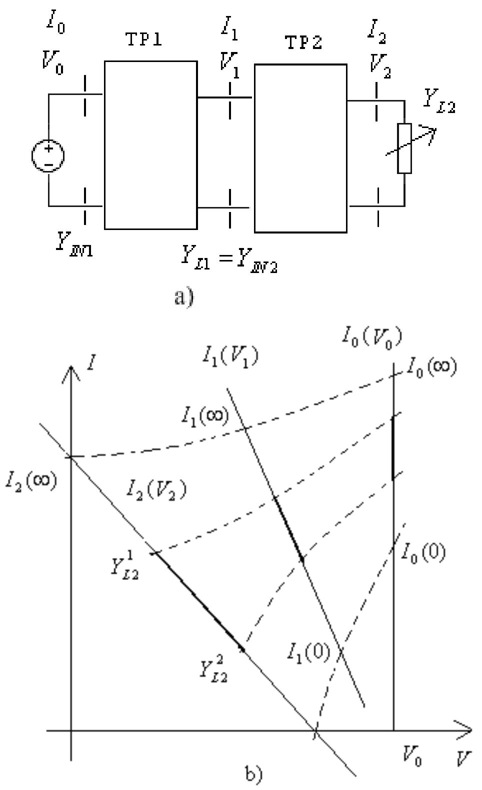


Fig.12. Cascade connection of two two-port –a) and correspondence of *I-V* characteristics –b)

If the mapping is viewed as the projective transformation, the invariant, which is the cross-ratio of four points, is performed and defines the same length of segments. Thus, networks of this kind are projectively – similar. Therefore, there is some the electro - biological analogy: disproportionate change of segments of load line for different parts of a network corresponds to growth changes of parts of the human body.

***Projective plane.*** If the network contains two changeable load, projective geometry for the plane is being shown. In this case, the load straight lines form the coordinate triangles. Therefore, networks of this kind are also projectively- similar.

## V. CONFORMAL TRANSFORMATIONS OF ELECTRIC NETWORK

Let us consider a power supply system with two voltage regulators $VR_1, VR_2$ and loads $R_1, R_2$ in Fig.13. The regulators define voltage transmission coefficient or transformation ratio $n_1, n_2$. The voltage regulators are connected to a limited capacity supply voltage source $U_0$. An interference of the regulators on regimes or load voltages $U_1, U_2$ is observed because of an internal resistance $R_i$. Let us consider the case $R_i = R_1 = R_2$. The network behaviour or "kinematics" via variable parameters $n_1, n_2$ is described by a sphere in the coordinates $U_1, U_2, U_i$ in Figs.3,4. For regulation, it is better to use such groups of transformations or movements of points in the planes $U_1, U_2$ and $n_1, n_2$, when it is impossible to deduce a working point over the circles, which correspond to the equator of sphere by finite switching number.

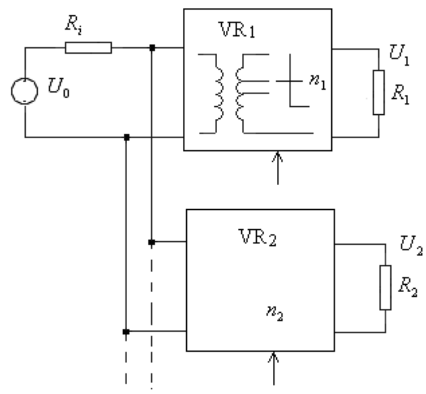


Fig.13. Power supply system with two voltage regulators

In this sense, we come to hyperbolic geometry. On the plane $U_1, U_2$ it is the Beltrami- Klein's model and on the plane $n_1, n_2$ it is the Poincare's model. The corresponding circle carries the name of the absolute and defines infinitely remote border.

Let us put the value $n_2 = 0$. Then, the regime change goes only on axes $U_1$ and $n_1$. The conformity of the characteristic points and running point is shown in Fig.14.

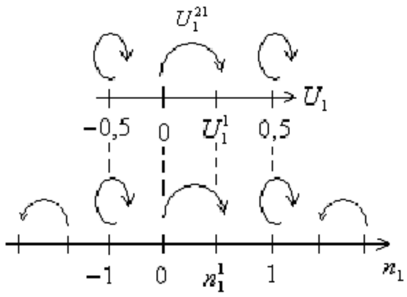


Fig.14. Conformity of the variables $U_1, n_1$ of the hyperbolic transformations

In the methodical sense, it is useful to notice this hyperbolic transformation by analogy, corresponds to the relativistic rule of speed composition in relative movement mechanics. If, for example $U_1^1 = 0.5$, then $U_1^2 = 0.5$ and it not depends from value $U_1^{21}$.

In the case $R_i \neq R_1 \neq R_2$, the sphere will be transformed to an ellipsoid. Therefore, these network will be conformally or Möbius- similar for the plane $n_1, n_2$.

Therefore, there is some the electro - biological analogy: the change of network parameters corresponds to growth changes of biological objects.

## CONCLUSION

- The analysis of human growth and analysis of operating regimes of electric networks shows an invariant of projective and conformal transformations.
- Different types of the cross-ratio take place for an electric network.
- The change of an operating regime of the given network or change of network parameters results to the projective or conformal similarity of networks.
- Established the electro - biological analogy develops a methodological basis of application of non-Euclidean geometries for these areas.

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