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The application of Mathematica to research the restricted eight bodies problem

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Abstract

Newton's restricted problem of eight bodies is investigated. In this paper an effective way of determining the stationary points of differential equations describing this problem is exposed. Analytic and numerical calculations are done with the system Mathematica.

Keywords: Newtonian problem, equation of motion, configuration, particular solution, equilibrium point.

1 Introduction

We consider the Newtonian problem of n bodies.This problem consists in studying the motion of n bodies in the Newtonian gravitational field. Its description is very simple [1], [2]. It is well-known that Newtonian many-body problem is not integrable in general.

The development of new computer technologies has provided the opportunity to otherwise approach the problem of n-bodies. Computer algebra system Mathematica is a very powerful tool for doing both symbolic and numeric calculations [4]. In many cases it turned out to be possible to construct not only approximate but exact solutions of differential equations of motion.

In this paper we study the existence of stationary solutions (the equilibrium positions) in the restricted eight bodies problem with incomplete symmetry, obtained with the help of symbolic calculus system Mathematica (SCS Mathematica).

2 Description of the configuration

Assume that in a non-inertial space P_0xyz there is the motion of eight bodies P_0, P_1, P_2, P_3, P_4, P_5, P_6, P with the masses m_0, m_1, m_2, m_3, m_4, m_5, m_6, μ, which attract each other in accordance with the law of universal attraction. We will investigate the planar dynamic pattern formed by a square, in the vertices of which the bodies P_1, P_2, P_3, P_4 are placed. The body P_0 is the center of the square and the bodies P_5, P_6 are placed on the diagonal P_1P_3 of the square at equal distances from P_0. We consider that $m_5 = m_6$ and the configuration rotates around the body P_0 with the constant angular velocity ω, which is determined from the model parameters. It will be studied the motion of the infinitely small mass μ (the so-called passive gravitational body with $\mu \approx 0$) in the gravitational field formed by the seven bodies P_0, P_1, P_2, P_3, P_4, P_5, P_6 that attract each other and attract the body P. As we study the planar configuration, we have $z_j = 0,\ j = 0, 1, ..., 7$. We can assume that $P_1(1,1)$, $P_2(-1,1)$, $P_3(-1,-1)$, $P_4(1,-1)$, $P_5(\alpha,\alpha)$, $P_6(-\alpha,-\alpha)$, $f = 1$, $m_0 = 1$, $m_5 = m_6$, then out of the differential equations of the motion the following conditions of existence of this configuration are obtained: $m_1 = m_3$, $m_2 = m_4 = f_1\,(m_1,\alpha)$, $m_5 = m_6 = f_2\,(m_1,\alpha)$, $\omega^2 = f_3\,(m_1,\alpha)$.

The differential equations of the motion of the point P in the gravitational field of the points P_0, P_1, P_2, P_3, P_4, P_5, P_6 in the rotating Cartesian coordinate system P_0xyz have the form [1]:

$$
\begin{cases}
\dfrac{d^2X}{dt^2} - 2\omega\dfrac{dY}{dt} = \omega^2 X - \dfrac{fm_0X}{r^3} + \dfrac{\partial R}{\partial X}, \\
\dfrac{d^2Y}{dt^2} + 2\omega\dfrac{dX}{dt} = \omega^2 Y - \dfrac{fm_0Y}{r^3} + \dfrac{\partial R}{\partial Y}, \\
\dfrac{d^2Z}{dt^2} = -\dfrac{fm_0Z}{r^3} + \dfrac{\partial R}{\partial Z},
\end{cases}
\tag{1}
$$

where

$$\begin{cases} R = f\sum\limits_{j=1}^{6} m_j \left(\dfrac{1}{\Delta_{kj}} - \dfrac{XX_j + YY_j + ZZ_j}{r_j^3} \right), \\ \Delta_j^2 = (X_j - X)^2 + (Y_j - Y)^2 + (Z_j - Z)^2, \\ r_j^2 = X_j^2 + Y_j^2 + Z_j^2,\ r^2 = X^2 + Y^2 + Z^2, \\ j = 1, 2, ..., 6. \end{cases} \tag{2}$$

According to the definition of the stationary solutions of the differential equations, the equilibrium positions (in case when they exist) are solutions of the functional system of equations:

$$\begin{cases} u = 0,\ v = 0,\ w = 0, \\ \omega^2 X + 2\omega v - \dfrac{fm_0 X}{r^3} + \dfrac{\partial R}{\partial X} = 0, \\ \omega^2 Y - 2\omega u - \dfrac{fm_0 Y}{r^3} + \dfrac{\partial R}{\partial Y} = -\dfrac{fm_0 Z}{r^3} + \dfrac{\partial R}{\partial Z} = 0, \end{cases} \tag{3}$$

or in the deployed form

$$\begin{cases} u = 0,\ v = 0,\ w = 0, \\ \omega^2 x + 2\omega v - \dfrac{fm_0 X}{r^3} - f\sum\limits_{j=1}^{6} m_j \left(\dfrac{X - X_j}{\Delta_j^3} + \dfrac{X_j}{r_j^3} \right) = 0, \\ \omega^2 y - 2\omega u - \dfrac{fm_0 Y}{r^3} - f\sum\limits_{j=1}^{6} m_j \left(\dfrac{Y - Y_j}{\Delta_j^3} + \dfrac{Y_j}{r_j^3} \right) = 0, \\ -\dfrac{fm_0 Z}{r^3} - f\sum\limits_{j=1}^{6} m_j \left(\dfrac{Z - Z_j}{\Delta_j^3} + \dfrac{Z_j}{r_j^3} \right) = 0, \end{cases} \tag{4}$$

$$\begin{cases} \Delta_j^2 = (X_j - X)^2 + (Y_j - Y)^2 + (Z_j - Z)^2, \\ r_j^2 = X_j^2 + Y_j^2 + Z_j^2,\ r^2 = X^2 + Y^2 + Z^2, \\ j = 1, 2, ..., 6. \end{cases} \tag{5}$$

For simplicity as above it has been taken $f = 1$, $m_0 = 1$. Replacing in relations (4) $(X_j, Y_j, Z_j = 0)$, $Z = 0$, $m_2 = m_4 = f_1(m_1, \alpha)$, $m_5 = m_6 = f_2(m_1, \alpha)$ and $\omega^2 = f_3(m_1, \alpha)$, determined above for admissible α and m_1, we obtain the following system:

$$\begin{cases} u = 0,\ v = 0,\ w = 0, \\ f(x,y) = \omega^2 x + 2\omega v - \dfrac{x}{(x^2+y^2)^{\frac{3}{2}}} + \\ +m_1\left(\dfrac{-1-x}{\left((1+x)^2+(1+y)^2\right)^{3/2}} + \dfrac{1-x}{\left((1-x)^2+(1-y)^2\right)^{3/2}}\right) + \\ +m_4\left(\dfrac{1-x}{\left((1-x)^2+(1+y)^2\right)^{3/2}} + \dfrac{-1-x}{\left((1+x)^2+(1-y)^2\right)^{3/2}}\right) + \\ +m_6\left(\dfrac{-\alpha-x}{\left((\alpha+x)^2+(\alpha+y)^2\right)^{3/2}} + \dfrac{\alpha-x}{\left((\alpha-x)^2+(\alpha-y)^2\right)^{3/2}}\right) = 0, \\ g(x,y) = \omega^2 y - 2\omega u - \dfrac{y}{(x^2+y^2)^{\frac{3}{2}}} + \\ +m_1\left(\dfrac{-1-y}{\left((1+x)^2+(1+y)^2\right)^{3/2}} + \dfrac{1-y}{\left((1-x)^2+(1-y)^2\right)^{3/2}}\right) + \\ +m_4\left(\dfrac{1-y}{\left((1-x)^2+(1+y)^2\right)^{3/2}} + \dfrac{-1-y}{\left((1+x)^2+(1-y)^2\right)^{3/2}}\right) + \\ +m_6\left(\dfrac{-\alpha-y}{\left((\alpha+x)^2+(\alpha+y)^2\right)^{3/2}} + \dfrac{\alpha-y}{\left((\alpha-x)^2+(\alpha-y)^2\right)^{3/2}}\right) = 0. \end{cases} \tag{6}$$

3 Determination of equilibrium points

The equations in the system (6) have a rather complicated structure. Its solving is quite cumbersome. If the solution of the system (6) is determined, then the solution of the equilibrium position of differential equations describing the restricted problem of the eight bodies is obtained. Using the graphical package of Mathematica for different parameter values α and m_1, the graphs of the curves $f(x, y)$ and $g(x, y)$ described by the equations in the system (6) have been constructed. Obviously, the points of intersection of these curves in the plain P_0xy will be the equilibrium positions of the investigated system. We will name the points that are on the lines passing through the center of the configuration and any peak of the square as radial equilibrium position (we will note them in the future by N_i). We will name the other points as equilibrium bisectorial positions (we will note them in the future by S_i). For this we use the following algorithm:

Algorithm 1

- constructs the configuration and graphs of the curves f and g;
- calculates the coordinates of the equilibrium bisectorial points S_i, $i = 1, ...4$ and displays them on the computer screen;
- shows the position of the point S_1.

This algorithm in the SCS Mathematica can be realised in the following way:

Algorithm 1 in SCS Mathematica

$graph[n\_, a\_] :=$
$Module[\{m_1 = n,\ \alpha = a\},\ gf = f(x, y, m_1, \alpha); gg = g(x, y, m_1, \alpha);$
$cpx = ContourPlot[gf, \{x,\ -2.5,\ 2.5\},\ \{y,\ 2.5,\ 2.5\},\ Contours \to \{0\},$
$ContourShading \to False, PlotPoints \to 100, ContourStyle \to$
$\{Black\},\ Axes \to True, Frame \to False];$
$cpy = ContourPlot[g, \{x,\ -2.5,\ 2.5\},\ \{y,\ -2.5,\ 2.5\},\ Contours \to \{0\},$
$ContourShading \to False, PlotPoints \to 100, ContourStyle \to$
$\{Dashed\},\ Axes \to True, Frame \to False];$
$square =\ ListPlot[\{\{1, 1\}, \{1, -1\}, \{-1, -1\}, \{-1, 1\}\},$
$PlotStyle \to \{PointSize[0.02]\}];$
$points =\ ListPlot[\{\{\alpha,\ \alpha\},\ \{-\alpha,\ -\alpha\}\},$
$PlotStyle \to \{PointSize[0.02]\}];$

$M0 := Graphics[Text[''P_0'', \{-0.15, -0.15\}]];$
$M1 := Graphics[Text[''P_1'', \{0.85, 1.05\}]];$
$M2 := Graphics[Text[''P_2'', \{-0.85, 1.05\}]];$
$M3 := Graphics[Text[''P_3'', \{-0.85, -1.05\}]];$
$M4 := Graphics[Text[''P_4'', \{0.85, -1.05\}]];$
$M5 := Graphics[Text[''P_5'', \{\alpha - 0.1, \alpha - 0.1\}]];$
$M6 := Graphics[Text[''P_6'', \{-\alpha + 0.1, -\alpha + 0.1\}]];$
$f1 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 1\}, \{y, 0\}];$
$S1 := Graphics[Text[''S_1'', \{1.55, -0.25\}]]; Print[''S_1'', f1];$
$f2 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 0\}, \{y, -1\}]; Print[''S_2'', f2];$
$f3 = FiindRoot[\{gf == 0, gg == 0\}, \{x, -1\}, \{y, 0\}]; Print[''S_3'', f3];$
$f4 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 0\}, \{y, 1\}]; Print[''S_4'', f4];$
$Show[points, square, cpx, cpy, p1, p2, M0, M1, M2, M3, M4, M5,$
$M6, S1,$
$PlotRange \to \{\{-2, 2\}, \{-2, 2\}\}, DisplayFunction \to$
$\$DisplayFunction,$
$AxesLabel \to \{x, y\}, AspectRatio \to Automatic,$
$PlotLabel \to'' m_1 ='' <> ToString[n];'' \alpha ='' <> ToString[a]'''']].$

For $m_1 = 0.01$ and $\alpha = 0.8584$ the result of this program is displayed in Figure 1.

4 Concluding remarks

In [3] A. Wintner introduced the concept of central configuration. The research configuration is of this type. The SCS Mathematica gives a possibility to consider various particular cases in the Newtonian problem of n bodies more effectively. The SCS Mathematica offers the opportunity to construct not only approximate but exact solutions of differential equations of motion. It is well known that for investigating the stability of stationary points, it is necessary to determine the equilibrium points of the configuration. In the present article we described an algorithm in the SCS Mathematica for determining the conditions of existence of equilibrium points in the restricted problem of eight bodies, and, in cases of equilibrium points' existence, the method for their building.

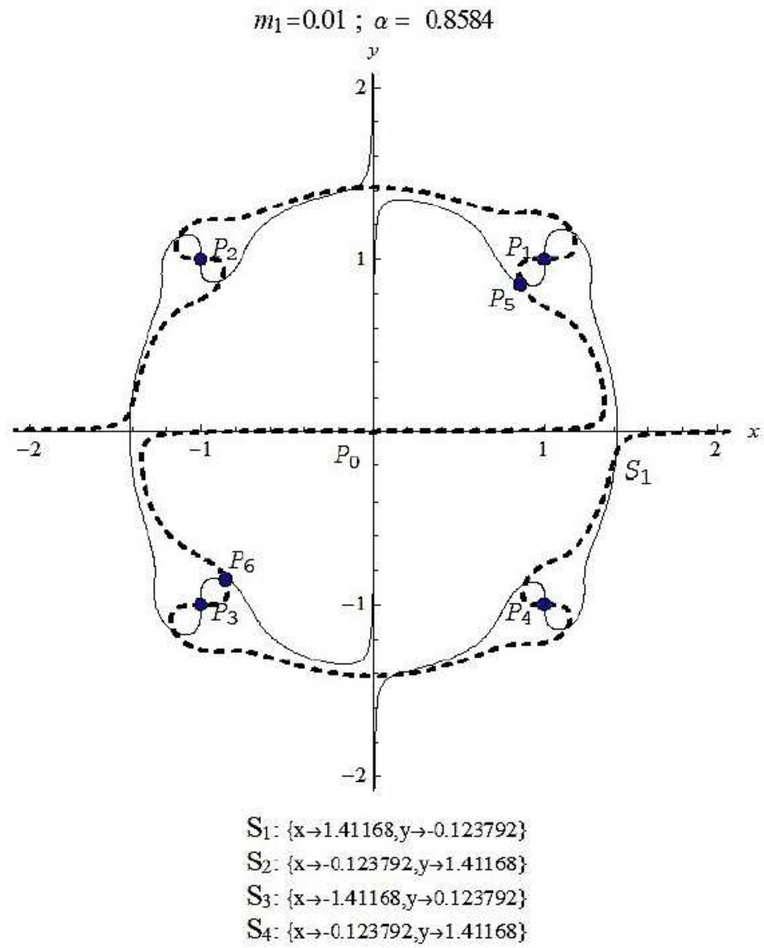

Figure 1. graph[0.01,0.8584]

References

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