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First integrals with polynomials not higher than second order of the mathematical model of the intrinsic transmission dynamics of tuberculosis

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Abstract

For the mathematical model of the intrinsic transmission dynamics of tuberculosis (TB) all first integrals with polynomials not higher than second order were found.

Keywords: tuberculosis, Lie algebra, first integral.

Consider three-dimensional autonomic real differential system which simulates the intrinsic transmission dynamics of tuberculosis [1], [2]

$$\frac{dS}{dt} = \tau - \mu S - \beta ST, \quad \frac{dL}{dt} = -\delta L - \mu L + (1-p)\beta ST,$$
$$\frac{dT}{dt} = \delta L - (\mu + \nu)T + p\beta ST. \tag{1}$$

The parameters of the system (1) are described in Table 1 (see page 258).

According to [3] we obtain

Theorem 1. *The system (1) admits the noncommutative Lie algebra of operators of the form*

$$X_1 = S\frac{\partial}{\partial S} + L\frac{\partial}{\partial L} + T\frac{\partial}{\partial T} + D_1, \quad X_2 = (-\frac{\tau}{\beta} - \frac{\nu}{\beta}S + ST)\frac{\partial}{\partial S} +$$
$$+[\frac{\delta-\nu}{\beta}L + (p-1)ST]\frac{\partial}{\partial L} - (\frac{\delta}{\beta}L + pST)\frac{\partial}{\partial T} + D_2, \tag{2}$$

where

$$D_1 = -\beta\frac{\partial}{\partial\beta} + \tau\frac{\partial}{\partial\tau}, \quad D_2 = (\mu+\nu)\frac{\partial}{\partial\beta} - \frac{\tau}{\beta}(\mu+\nu)\frac{\partial}{\partial\tau}, \qquad (3)$$

and the structural equation is $[X_1, X_2] = X_2$.

Table 1. Variables and parameters of the sistem (1)

Value	Description
$S(t)$	number of sensible persons in the moment t
$L(t)$	number of infected persons in the moment t
$T(t)$	number of infectious persons in the moment t
$\beta T(t)$	force of infection per capita in the moment t
τ	influx of young people
μ	average mortality from causes not related to TB
p	probability of rapid progression of the disease
δ	constant of speed of reactivation of TB infection
ν	additional mortality caused by active TB
β	transfer coefficient of TB infection

Note that the expressions

$$U_1 = \beta\tau, \quad U_2 = \mu, \quad U_3 = \nu, \quad U_4 = \delta, \quad U_5 = p, \qquad (4)$$

are invariants of the system (1) with respect to the operators (2)–(3), i.e. $D_1(U_i) = D_2(U_i) = 0$ $(i = \overline{1,5})$.

Further we assume that U_i $(i = \overline{1,4})$ from (4) do not vanish. This guarantees us the existence of the quadratic part ST and of the free term τ in the system (1). The condition $\mu\nu\delta \neq 0$ arises from the medical sense of the parameters.

We determine the coordinates of the vector $(\tau, \beta, \mu, \delta, \nu, p)$ which contain the parameters of the system (1) when the invariants U_i $(i = \overline{1,4})$ (see (4)) are different from zero and first integral has the form

$$I_q(S, L, T, t) = P_q(S, L, T)\exp(\lambda t) \ (q \leq 2). \qquad (5)$$

Assume that

$$P_q(S,L,T) = a + bS + cL + dT + eS^2 + fL^2 + gT^2 + \\ +2hSL + 2kST + 2lLT. \quad (6)$$

The coefficients of the polynomial (6) and the parameter λ are real unknown. From $\frac{dI_q}{dt} \equiv 0$ ($q \le 2$) using the relations (5)–(6) under system (1) we obtain the following system of polynomial equations:

$$\lambda a + \tau b = 0,\ 2\tau e + (\lambda - \mu)b = 0,\ 2\tau h - \mu c + \delta(d - c) + \lambda c = 0,$$

$$2\tau k + (\lambda - \mu - \nu)d = 0,\ (\lambda - 2\mu)e = 0,\ -2\mu f + 2\delta(l - f) + \lambda f = 0,$$

$$(\lambda - 2\mu - 2\nu)g = 0,\ 2\mu h + \delta(h - k) - \lambda h = 0,\ (2\mu + \nu)l - \delta(g - l) - \lambda l = 0,$$

$$\beta(-b + c - cp + dp) + 2(\lambda - 2\mu - \nu)k = 0,\ \beta(e - h + hp - kp) = 0,$$

$$\beta(k - l - gp + lp) = 0,\ \beta(f - h - fp + lp) = 0. \quad (7)$$

Consequently we arrive at the next result

Theorem 2. *Assume that the conditions* $U_1U_2U_3U_4 \neq 0$ *and* $0 \le U_5 \le 1$ *hold. Then the system (1) possessing the vector* $(\tau, \beta, \mu, \delta, \nu, p)$ *has* 5 *first integrals of the form (5)–(6) (see Table 2).*

Table 2. First integrals of the sistem (1)

$(\tau, \beta, \mu, \delta, \nu, p)$	**First integral**
$(\tau, \beta, \mu, p\nu, \nu, p)$	$I_1^{(1)} = (L + \frac{p-1}{p} T)\exp(t(\mu + \nu))$
$(\tau, \beta, \mu, \delta, \nu, 1)$	$I_1^{(2)} = L\exp(t(\delta + \mu))$
$(\tau, \beta, \mu, -\mu, \nu, 1)$	$I_2^{(1)} = a + L(c + fL)$
$(\tau, \beta, \mu, -p\mu, -\mu, p)$	$I_2^{(2)} = a + (L + \frac{p-1}{p} T)(c + f(L + \frac{p-1}{p} T))$
$(\tau, \frac{\mu(\nu^2 - \mu^2)}{\nu\tau}, \mu, -\nu, \nu, 0)$	$I_2^{(3)} = ((\nu^2 - \mu^2)((L+S)^2 + 2T(L+S))/(2\mu\tau) + +(L + S + T) + T\nu/\mu - S\nu^2/\mu^2 - -\tau/(2\mu) + \nu^2\tau/(2\mu^3))\exp(2t\mu)$

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