Minimal generating set and properties of commutator of Sylow subgroups of alternating and symmetric groups

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Summary. Given a permutational wreath product sequence of cyclic groups [12, 6] of order 2 we research a commutator width of such groups and some properties of its commutator subgroup. Commutator width of Sylow 2-subgroups of alternating group A_{2^k}, permutation group S_{2^k} and $C_p \wr B$ were founded. The result of research was extended on subgroups $(Syl_2 A_{2^k})'$, $p > 2$. The paper presents a construction of commutator subgroup of Sylow 2-subgroups of symmetric and alternating groups. Also minimal generic sets of Sylow 2-subgroups of A_{2^k} were founded. Elements presentation of $(Syl_2 A_{2^k})'$, $(Syl_2 S_{2^k})'$ was investigated. We prove that the commutator width [14] of an arbitrary element of a discrete wreath product of cyclic groups C_{p_i}, $p_i \in \mathbb{N}$ is 1.

Let G be a group. The commutator width of G, $cw(G)$ is defined to be the least integer n, such

that every element of G' is a product of at most n commutators if such an integer exists, and $cw(G) = \infty$ otherwise. The first example of a finite perfect group with $cw(G) > 1$ was given by Isaacs in [9].
A form of commutators of wreath product $A \wr B$ was briefly considered in [7]. For more deep description of this form we take into account the commutator width ($cw(G)$) which was presented in work of Muranov [14]. This form of commutators of wreath product was used by us for the research of $cw(Syl_2 A_{2^k})$, $cw(Syl_2 S_{2^k})$ and $cw(C_p \wr B)$. As well known, the first example of a group G with $cw(G) > 1$ was given by Fite [4]. We deduce an estimation for commutator width of wreath product $B \wr C_p$ of groups C_p and an arbitrary group B taking into the consideration a $cw(B)$ of passive group B.
A research of commutator-group serves to decision of inclusion problem [5] for elements of $Syl_2 A_{2^k}$ in its derived subgroup $(Syl_2 A_{2^k})'$.
Results. We consider $B \wr (C_p, X)$, where $X = \{1, .., p\}$, and $B' = \{[f, g] \mid f, g \in B\}$, $p \geq 1$. If we fix some indexing $\{x_1, x_2, ..., x_m\}$ of set the X, then an element $h \in H^X$ can be written as $(h_1, ..., h_m)$ for $h_i \in H$.

The set X^* is naturally a vertex set of a regular rooted tree, i.e. a connected graph without cycles and a designated vertex v_0 called the root, in which two words are connected by an edge if and only if they are of form v and vx, where $v \in X^*$, $x \in X$. The set $X^n \subset X^*$ is called the n-th level of the tree X^* and $X^0 = \{v_0\}$. We denote by $v_{j,i}$ the vertex of X^j, which has the number i. Note that the unique vertex $v_{k,i}$ corresponds to the unique word v in alphabet X. For every automorphism $g \in AutX^*$ and every word $v \in X^*$ define the section (state) $g_{(v)} \in AutX^*$ of g at v by the rule: $g_{(v)}(x) = y$ for $x, y \in X^*$ if and only if $g(vx) = g(v)y$. The subtree of X^* induced by the set of vertices $\cup_{i=0}^k X^i$ is denoted by $X^{[k]}$. The restriction of the action of an automorphism $g \in AutX^*$ to the subtree $X^{[l]}$ is denoted by $g_{(v)}|_{X^{[l]}}$. A restriction $g_{(v)}|_{X^{[1]}}$ is called the vertex permutation (v.p.) of g in a vertex v.

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The commutator length of an element g of the derived subgroup of a group G is denoted *clG(g)*, is the minimal n such that there exist elements $x_1, ..., x_n, y_1, ..., y_n$ in G such that $g = [x_1, y_1]...[x_n, y_n]$. The commutator length of the identity element is 0. The commutator width of a group G, denoted $cw(G)$, is the maximum of the commutator lengths of the elements of its derived subgroup $[G, G]$.

Let us make some notations. The commutator of two group elements a and b, denoted

$$[a, b] = aba^{-1}b^{-1},$$

conjugation by an element b as

$$a^b = bab^{-1},$$

$\sigma = (1, 2, \ldots, p)$. Also $G_k \simeq Syl_2 A_{2^k}$, $B_k = \wr_{i=1}^k C_2$. The structure of G_k was investigated in [6]. For this research we can regard G_k and B_k as recursively constructed i.e. $B_1 = C_2$, $B_k = B_{k-1} \wr C_2$ for $k > 1$, $G_1 = \langle e \rangle$, $G_k = \{(g_1, g_2)\pi \in B_k \mid g_1 g_2 \in G_{k-1}\}$ for $k > 1$.

The following Lemma follows from the corollary 4.9 of the Meldrum's book [7].

Lemma 1. *An element of form $(r_1, \ldots, r_{p-1}, r_p) \in W' = (B \wr C_p)'$ iff product of all r_i (in any order) belongs to B', where B is an arbitrary group.*

Proof. Analogously to the Corollary 4.9 of the Meldrum's book [7] we can deduce new presentation of commutators in form of wreath recursion

$$w = (r_1, r_2, \dots, r_{p-1}, r_p),$$

where $r_i \in B$. □

Lemma 2. *For any group B and integer $p \geq 2$, $p \in \mathbb{N}$ if $w \in (B \wr C_p)'$ then w can be represented as the following wreath recursion*

$$w = (r_1, r_2, \dots, r_{p-1}, \prod_{j=1}^{k} [f_j, g_j]),$$

where $r_1, \dots, r_{p-1}, f_j, g_j \in B$, and $k \leq cw(B)$.

Lemma 3. *An element $(g_1, g_2)\sigma^i \in G'_k$ iff $g_1, g_2 \in G_{k-1}$ and $g_1 g_2 \in B'_{k-1}$.*

Lemma 4. *For any group B and integer $p \geq 2$ inequality*

$$cw(B \wr C_p) \leq \max(1, cw(B))$$

holds.

Corollary 1. *If $W = C_{p_k} \wr \dots \wr C_{p_1}$ then for $k \geq 2$ $cw(W) = 1$.*

Corollary 2. *Commutator width $cw(Syl_p(S_{p^k})) = 1$ for prime p and $k > 1$ and commutator width $cw(Syl_p(A_{p^k})) = 1$ for prime $p > 2$ and $k > 1$.*

Theorem 1. *Elements of $Syl_2 S'_{2^k}$ have the following form $Syl_2 S'_{2^k} = \{[f, l] \mid f \in B_k, l \in G_k\} = \{[l, f] \mid f \in B_k, l \in G_k\}$.*

For the group $G_k^{''}$ we denote by s_{ij} vertex permutation of automorphism in v_{ij}.

Lemma 5. *The group $G_k^{''}$ has equal permutation in vertices of X^2, viz $s_{21} = s_{22} = s_{23} = s_{24}$.*

Theorem 2. *Commutator width of the group $Syl_2 A_{2^k}$ equal to 1 for $k \geq 2$.*

Proposition 1. *The subgroup $(syl_2 A_{2^k})'$ has a minimal generating set of $2k - 3$ generators.*

Conclusion . The commutator width of Sylow 2-subgroups of alternating group A_{2^k}, permutation group S_{2^k} and Sylow p-subgroups of $Syl_2 A_p^k$ ($Syl_2 S_p^k$) is equal to 1. Commutator width of permutational wreath product $B \wr C_n$, were B is an arbitrary group, was researched.

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