Units in generalized derivatives of quasigroups

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Here we mainly use [1, 10].

Definition 1. *An n-ary groupoid* (Q, A) *with n-ary operation A such that in the equality* $A(x_1, x_2, \dots, x_n) = x_{n+1}$ *the fact of knowing any n elements of the set* $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ *uniquely specifies the remaining one element, is called an n-ary quasigroup* [3].

If we put $n = 2$, then we obtain one more definition of a binary quasigroup.

Definition 2. *From Definition 1 it follows that with a given binary quasigroup* (Q, A) *it is possible to associate* $(3!-1)$ *other so-called parastrophes of quasigroup* (Q, A)*:*

1. $A(x_1, x_2) = x_3 \Longleftrightarrow$
2. $A^{(12)}(x_2, x_1) = x_3 \Longleftrightarrow$
3. $A^{(13)}(x_3, x_2) = x_1 \Longleftrightarrow$
4. $A^{(23)}(x_1, x_3) = x_2 \Longleftrightarrow$
5. $A^{(123)}(x_2, x_3) = x_1 \Longleftrightarrow$
6. $A^{(132)}(x_3, x_1) = x_2$

[11, p. 230], [1, p. 18].

The following table (using Table 1) shows for each kind of translation the equivalent one in each of the (six) parastrophes of a quasigroup $(Q, \cdot)$. In fact, Table 1 is a rewritten form of results on three kinds of translations from [2]. See also [4, 9].

From Table 1 it follows, for example, that $R^{(132)} = L^{-1} = L^{(23)} = P^{(13)} = (R^{-1})^{(12)} = (P^{-1})^{(123)}$.

By the letter T we denote the set of all quasigroup translations of a fixed quasigroup $(Q, \cdot)$ and their inverses relatively one fixed element, say, relatively an element a.

Definition 3. *Quasigroup* $(Q, \star) = (Q, \cdot)(\alpha, \beta, \gamma)$*, where* $(Q, \star)$ *is isostrophic image of quasigroup* $(Q, \cdot)$*, i.e.,* $\cdot \in \{A, A^{(12)}, A^{(13)}, A^{(23)}, A^{(123)}, A^{(132)}\}$*,* $\alpha, \beta, \gamma \in$

Table 1. Translations of quasigroup parastrophes.

	ε	(12)	(13)	(23)	(123)	(132)
R	R	L	R^{-1}	P	P^{-1}	L^{-1}
L	L	R	P^{-1}	L^{-1}	R^{-1}	P
P	P	P^{-1}	L^{-1}	R	L	R^{-1}
R^{-1}	R^{-1}	L^{-1}	R	P^{-1}	P	L
L^{-1}	L^{-1}	R^{-1}	P	L	R	P^{-1}
P^{-1}	P^{-1}	P	L	R^{-1}	L^{-1}	R

T, and in every case one of the translations α, β, γ is an identity permutation, is called an isostrophic (generalized) derivative of quasigroup $(Q, \cdot)$ with respect to element a.

We research all 1944 cases when a generalized derivative of a quasigroup has a unit [5]. We actively used Prover and Mace [8, 7].

Example 1. *We demonstrate that in general, isostrophical derivative of the form:*

$$x \diamond y = L_a x \backslash P_a^{-1} y \tag{1}$$

has no left, right, middle unit. Using table of translations (Table 1), we re-write this isostrophical derivative in the following form $x \diamond y = (a \cdot x)\backslash(a/y)$.

We can say that this isostrophical derivative $(Q, \diamond)$ has no left (right, middle) identity element for a fixed element $a \in Q$. Indeed, we take the following quasigroup $(Q, \cdot)$ (the cyclic group of order 3)

$\cdot$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and $a := 0$. Its isostrophical image $(Q, \diamond)$ of the form (1) has no left and right unit

$\diamond$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

We take the following quasigroup $(Q, \cdot)$

$\cdot$	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

and $a := 0$*. Its isostrophical image* $(Q, \diamond)$ *of the form (1) has no middle unit*

$\diamond$	0	1	2
0	1	2	0
1	2	0	1
2	0	1	2

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