General form of autotopies of 3-IP-loop

URSU Leonid

1. Introduction

Definition 1. *Ternary loop $Q(A)$ with identity element e is called a loop with inverse property (3-IP-loop) if on the set Q there exists permutations (bijections) I_{ij} $i, j \in \overline{1,3}$ such that the following identity is true*

$$A(\{I_{ij}x_j\}_{j=1}^{i-1}, A(x_1^3), \{I_{ij}x_j\}_{j=i+1}^{3}) = x_i \tag{1}$$

for every $x_1^3 \in Q^3$, where $I_{ii} = I_{i4} = \varepsilon$.

The matrix

$$\begin{bmatrix} I_{ij} \end{bmatrix} = \begin{bmatrix} \varepsilon & I_{12} & I_{13} & \varepsilon \\ I_{21} & \varepsilon & I_{23} & \varepsilon \\ I_{31} & I_{32} & \varepsilon & \varepsilon \end{bmatrix}$$

is called a matrix of inversability for 3-IP-quasigroup, and permutations $I_{i,j}$ are called *permutations of inversability*, that are defined by the equalities

$$(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{3-j}{e}) = e \tag{2}$$

for every $x \in Q$ where $i, j \in \overline{1,3}$ [1].

Definition 2. *Element e is called a unit (identity element) of 3-loop $Q(A)$, if $A(e,e,x) = A(e,x,e) = A(x,e,e) = x$ for any $x \in Q$.*

Definition 3. *Row $(\alpha, \beta, \gamma, \delta)$ of permutations of the set Q is called autotopy of loop $Q(A)$, if $\delta^{-1}A(\alpha x, \beta y, \gamma z) = A(x, y, z)$ for every $x, y, z \in Q$. It is denoted as $A^T = A$.*

Notice we denote as $\mathfrak{A}_A$ the set of all autotopies of loop $Q(A)$.

It is known [2] that if $(\alpha, \beta, \gamma, \delta) \in \mathfrak{A}_A$, then $(\delta, I_{12}\beta I_{12}, I_{13}\gamma I_{13}, \alpha)$, $(I_{12}\alpha I_{12}, \delta, I_{23}\gamma I_{23}, \beta)$, $(I_{13}\alpha I_{13}, I_{23}\beta I_{23}, \delta, \gamma) \in \mathfrak{A}_A$.

Let $T = (\alpha, \beta, \gamma, \delta)$ be an autotopy of loop $Q(A)$ with unit e and matrix of inversability $[I_{ij}]$. Then $T_1 = (I_{12}\alpha I_{12}, \delta, I_{23}\gamma I_{23}, \beta) \in \mathfrak{A}_A \Rightarrow T_2 = T_1 T^{-1} = (I_{12}\alpha I_{12}\alpha^{-1}, \delta\beta^{-1}, I_{23}\gamma I_{23}\gamma^{-1}, \beta\delta^{-1}) \in \mathfrak{A}_A$.

We denote $(\alpha e, \beta e, \gamma e) = (\overline{k})$. Then from $(\alpha x, \beta y, \gamma z) = \delta(x, y, z)$ for $x = z = e$ we have $\delta y = (\alpha e, \beta y, \gamma e) = L_2(\overline{k})\beta y \Rightarrow \beta\delta^{-1} y = L_2^{-1}(\overline{k})y$.

Thus $T_2 = (I_{12}\alpha I_{12}\alpha^{-1}, L_2(\overline{k}), I_{23}\gamma I_{23}\gamma^{-1}, L_2^{-1}(\overline{k})) \Rightarrow$
$T_3 = (L_2^{-1}(\overline{k}), I_{12}L_2(\overline{k})I_{12}, I_{13}I_{23}\gamma I_{23}\gamma^{-1}I_{13}, I_{12}\alpha I_{12}\alpha^{-1}) \in \mathfrak{A}_A \Rightarrow$
$T_4 = T_3^{-1} = (L_2^{-1}(\overline{k}), L_1(\overline{k_{1,2}}), I_{13}\gamma I_{23}\gamma^{-1}I_{23}I_{13}, \alpha I_{12}\alpha^{-1}I_{12}) \in \mathfrak{A}_A$.
For $n = 2$ we have $T_4 = (L_u, R_u, L_u R_u)$, $u = \alpha e$.

Definition 4. *The elements $\alpha e, \gamma e \in Q$ for which $T_4 \in \mathfrak{A}_A$ are called Moufang elements.*

Definition 5. *If identity T_4 satisfies any $x_1^3 \in Q^3$ and for any $\alpha e = u$, $\beta e = v$, then loop $Q(A)$ is called Moufang loop.*

Definition 6. *See [3]. Permutation φ_i^j on the set Q is called (i, j)-pseudo-automorphism of n-quasigroup $Q(A)$, if on the set Q there exists a fixed row of elements $a_1^{i-1}, a_{i+1}^n \in Q$ such that*

$$A(a_1^{i-1}, \varphi_i^j(x_1^n), a_{i+1}^n) = A((\varphi_i^j(x_k))_{k=1}^{j-1}, A(a_1^{i-1}, \varphi_i^j(x_j), a_{i+1}^n), (\varphi_i^j(x_k))_{k=j+1}^n).$$

Fixed row of elements $a_1^{i-1}, a_{i+1}^n \in Q$ is called companion of pseudo-automorphism φ_i^j.

2. Main results

Theorem 1. *If $T = (\alpha, \beta, \gamma, \delta)$ is an autotopy of 3-IP-loop $Q(A)$, $\alpha e = \gamma e = e$ and $\alpha = \gamma$, then α is $(1-2)$-pseudo-automorphism of this loop with companions βe and $\gamma e = e$, i.e., $T_5 = (\alpha, L_1(\overline{k})\alpha, \alpha, L_1(\overline{k})\alpha) \in \mathfrak{A}_A$.*

Theorem 2. *Any autotopy of 3-IP-loop $Q(A)$ can be expressed as the product of two autotopies: one from which define Moufang elements but second define pseudo-automorphism of this loop*

$$T_6 = T_4 \cdot T_5 =$$
$$= (L_2(\overline{k}), L_1(\overline{k_{1,2}}), I_{13}I_{23}\gamma^{-1}I_{23}I_{13}, \alpha I_{21}\alpha^{-1}I_{21})(\varphi, L_1(\overline{k})\varphi, \varphi, L_1(\overline{k})\varphi),$$

where $\varphi = L_2^{-1}(\overline{k})\alpha$. It is true and inverse: every four permutations of the set Q of the type T_6 is an autotopy of 3-IP-loop $Q(A)$.

References

[1] V.D. Belousov, *n*-ary quasigroups. Izdat. "Ştiinţa", Kishinev, 1972, 227 pp., (Russian).

[2] L. A. Ursu. *n*-ary loops with the invertibility property. Mat. Issled. No. 113, Kvazigruppy i ikh Sistemy (1990), 108–118, (Russian).

[3] L. A. Ursu. Derivative operations and pseudoautomorphisms of *n*-ary quasigroups. Quasigroups and Latin squares. Mat. Issled. No. 71 (1983), 126–135, (Russian).

(URSU Leonid) Technical University of Moldova, Chişinău, Republic of Moldova
E-mail address: matematica@mail.utm.md