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The first boundary value problem for the nonlinear equation of heat conduction with deviation of the argument

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***Abstract* - The initial-boundary problem for the heat conduction equation with the inversion of the argument are considered. The Green's function of considered problem are determined. The theorem about the Poisson integral limitation is proved. The theorem declared that the Poisson integral determine the solution of the first boundary problem considered and proved.**

Keywords — heat nonlinear equation, boundary value problem, Green function; deviation argument

Introduction

In this paper the solution of the first boundary value problem for the heat conduction equation with the inversion of the argument is found. The process of heat conduction was first described by Jean Baptiste Joseph Fourier (1768 - 1830) in 1807 in the work "Equations with partial derivatives for heat conduction in solids". A description of the results of other scientists who studied and developed this theory is presented in [1]. Thermal conductivity is the molecular distribution of thermal energy in various solids, liquids and gases due to the difference in temperature and due to the fact that the particles are in direct contact with each other. Based on different criteria, models of heat conduction processes are divided into two groups of models using integral and fractional order derivatives. Models with derivatives of an integer order are stationary and non-stationary.

Non-stationary models of one-dimensional heat conduction are described by the equation of heat conduction [4]. Different methods of solving this problem are described in [2, 3, 4]. Applied aspects of such problems are described in [5, 6, 7].

If the evolution of the concentration of impurities, point defects, and the temperature field is studied, then the corresponding transfer coefficients are not constant values. Processes with spatially dependent transfer coefficients or the desired field are well studied (see, for example, [2]) and sufficiently describe processes in heterogeneous and nonlinear media [8]. Also described here are applied problems for modeling and research of which the transfer coefficients depend on the time change. At the same time, physically adequate modeling of situations most often requires research in a semi-limited area [7].

Our work examines the first boundary value problem for the non-homogeneous nonlinear heat conduction equation with argument deviation, which generalizes the corresponding problem from [7].

Formulation of the First Boundary Value Problem

Let $a>0, b>0$ be real numbers; $x\in R^{+}, t\in R^{+}$, are independent variables; f, φ, μ are known continuous functions; $u(x, t)$ is the desired function that describes the evolution of the system defined on the semi-axis $x\in R^{+}$ for all $t\in R^{+}$,. We will study the problem

$$u_t = a^2 u_{xx} + f(x,t,u(x,t-h)), x>0, t>h\,, \qquad (1)$$

$$u(x,t)|_{0\le t\le h} = \varphi(x,t), x\ge 0\,, \qquad (2)$$

$$u(0,t) = \mu(t), t\ge h \qquad (3)$$

which is the first boundary value problem, where the functions $\varphi(x,t)\in C(R^{+}\times\{0\le t\le h\})$ is initial function, $\mu(t)\in C(R_h^{+})$ is boundary function, $R_h^{+}\equiv\{t; t\ge h\}$, $R^{+}\equiv\{x; x\ge 0\}$), $f(x,t,u)\in C(R^{+}\times R_h^{+}\times R)$ is the inhomogeneity of equation (1) is well known. If a smooth solution of the problem (1) – (3) is sought up to the limit, then the initial and marginal functions must be consistent $\varphi(0,h)=\mu(h)$.

THE STEPS METHOD

Let $h \le t \le 2h$, then $t-h \ge 0$, $u(x,t-h)=\varphi(x,t)$. $x > 0$. Then in (1) – (3) we get the problem:

$$u_t = a^2 u_{xx} + f(x,t,\varphi(x,t)), x>0, h<t<2h, \quad (4)$$

$$u(x,t)|_{t=h} = \varphi(x,h), x \ge 0, \quad (5)$$

$$u(0,t)=\mu(t), t \ge h \quad (6)$$

with the conditions of agreed $\varphi(0,h)=\mu(h)$.

We are looking for a solution of problem (4) – (6) in the form of sum of three functions

$$u(x,\ t) = u_1(x,t) + u_2(x,\ t) + u_3(x,\ t), \quad (7)$$

where $u_i, 1 \le i \le 3$, respectively, take into account the influence only initial condition, the boundary condition and the inhomogeneity of the, that is, they are the solutions of such problems.

Problem 1. Find a function $u_1(x,\ t)$ that satisfies the conditions

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, x>0, h<t<2h, \quad (8)$$

$$u(x,h)=\varphi(x,h), x \ge 0, \quad (9)$$

$$u(0,t)=0, h \le t \le 2h, \quad (10)$$

moreover, $\varphi(0,h)=u(0,h)=0$ is a condition of agreed.

Problem 2. Find a function $u_2(x,\ t)$ that satisfies equation (8) and conditions

$$u(x,h)=0, x \ge 0, \quad (11)$$

$$u(0,t)=\mu(t), h \le t \le 2h, \quad (12)$$

moreover, $\mu(h)=0$ is a condition of agreed.

Problem 3. Find the function $u_3(x,\ t)$ that satisfies equation (1) and conditions (10), (11), which are agreed.

2.1. Solving problem 1

Let's expand the domain of definition of equation (8) and the initial condition to $x \in R, h \le t \le 2h$ and solve it with help of separation of variables method ($\tilde{u}_1(x,t)=X(x)T(t)$ and after rearrangement in (8) and separation of variables, we obtain that $T(t)=C(\lambda)e^{-\lambda^2 a^2 (t-h)}, X(x)=e^{i\lambda x}$, where λ is the variable separation parameter. Then the solution is $u_1(x,t,\lambda)=C(\lambda)e^{-\lambda^2 a^2 (t-h)+i\lambda x}, \lambda \in R$ and to take into account all $\lambda \in R$ \$ we create a function

$$\tilde{u}_1(x,t)=\int_{-\infty}^{\infty} C(\lambda)e^{-\lambda^2 a^2 (t-h)+I\lambda x} d\lambda, x \in R, h \le t \le 2h,$$

which satisfies condition (9). Then we get that

$$\varphi(x,h)=\int_{-\infty}^{\infty} C(\lambda)e^{i\lambda x} d\lambda \quad ,$$

and

$$C(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\xi,h)e^{-i\lambda\xi} d\xi \lambda \in R$$

and

$$\tilde{u}_1(x,t)=\int_{-\infty}^{\infty} \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{-\lambda^2 a^2 (t-h)+i\lambda(x-\xi)} d\lambda\right\}\varphi(\xi,h)d\xi.$$

The inner integral calculated in [4]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\lambda^2 (t-h)a^2 + i\lambda(x-\xi)} d\lambda = \frac{1}{2\sqrt{\pi(t-h}} e^{-\frac{(x-\xi)^2}{4a^2(t-h)}}$$

is denoted by $G(x-\xi; t-h)$ and is the fundamental solution of equation (8). Then

$$\tilde{u}_1(x,t)=\int_{-\infty}^{\infty} G(x0\xi; t-h)\varphi(\xi,h)d\xi, x \in R, h<t<2h. \quad (13)$$

We use formula (13) to construct a solution to problem 1. For this, instead of equation (8), we consider equations

$$\frac{\partial U(x,t)}{\partial t} = a^2 \frac{\partial^2 U(x,t)}{\partial x^2}, x \in R, t>h, \quad (14)$$

with conditions (9), (10), extending in condition (9) the initial function $\varphi(x,h)$ for $x < 0$ nonearly, and we leave condition (10) unchanged:

$$U(x,h)=\Psi(x,h)=\begin{cases}\varphi(x,h), x \ge 0,\\ -\varphi(-x,h), x<0.\end{cases} \quad (15)$$

Then, according to formula (13), the solution of problem (14), (15), (10) is

$$U(x,\ t)=\int_{-\infty}^{\infty} \{G(x-\xi; t-h) - G(x+\xi; t-h)\}\varphi(\xi,h)d\xi.$$

In the integral where $\xi < 0$, we replaced $\xi = -\xi$. Simplifying the difference of the exponents included in the expression for the function *G*, we obtain that

$$u_1(x,t)=\frac{1}{\sqrt{\pi(t-h)}}\int_0^{\infty} \varphi(x,h)e^{-\frac{x^2+\xi^2}{4a^2(t-h)}} sh\frac{x\xi}{2(t-h)} d\xi,$$

where $x > 0$, $h < t < 2h$. Using the method of mathematical induction, we prove that in case $x \ge 0, kh < t < (k+1)h$ the solution to problem 1 takes the form

$$u_1(x,t)=\frac{1}{\sqrt{\pi(t-kh)}}\int_0^{\infty} \varphi(\xi,kh)e^{-\frac{x^2+\xi^2}{4a^2(t-kh)}} sh\frac{x\xi}{2a^2(t-kh)} d\xi \quad (16)$$

Therefore, the following theorem is true.

Let's mark

$$G_1(x,y,t-kh)=\frac{1}{\sqrt{\pi(t-kh)}} e^{-\frac{x^2+y^2}{4a^2(t-kh)}} sh\frac{xy}{2a^2(t-kh)}, \quad (17)$$

$x \ge 0, y > 0, kh < t < (k+1)h, k \in N$.

Definition. A function $G_1(x,y,t-kh)$ is called a Green's function of problem (1), (2), (3) if it satisfies the following conditions:

1) the function $G_1(x,y,t-kh)$ is continuous on *x*, *y*, *t*, continuously differentiable on *t* and twice continuously differentiable on *x*, *y* when $x > 0$, $y > 0$, $kh < t < (k+1)h$, $k \in$ N, and possibly with the exception in the point $x = y$, $t = kh$;

2) the function $G_1(x,y,t-kh)$ by variables *x* and *y* satisfies the equation $\frac{\partial G_1}{\partial t} = a^2 \frac{\partial^2 G_1}{\partial x^2}$ everywhere except in the points $x = y$, $t = kh$, $k \in$ N;

3) the function $G_1(x,y,t-kh)$ satisfies the boundary condition $G_1(0, y, t - kh) = 0$.

The Green's function satisfying this definition is constructed above and takes the form (17)

$G_1(x,y,t-kh) = G_1(y,x,t-kh)$.

2.2. Properties of the solution of the problem 1

Given that

$$G_1(x,y,t-kh) = \frac{1}{2\sqrt{\pi(t-kh)}}\left\{e^{-\frac{(x-\xi)^2}{4a^2(t-kh)}} - e^{-\frac{(x+\xi)^2}{4a^2(t-kh)}}\right\}$$

we get from (16), when $|\varphi(\xi,kh)| \le M$

$$|u_1(x,t) \le M\frac{1}{2\sqrt{\pi(t-h)}}\left\{\int_0^\infty e^{-\frac{(x-\xi)^2}{4a^2(t-kh)}}d\xi - \int_0^\infty e^{-\frac{(x+\xi)^2}{4a^2(t-kh)}}d\xi\right\} \equiv$$

$\equiv M\{I_1 - I_2\}$.

In the integral I_1 we will do replacement $\alpha = \frac{\xi - x}{2\sqrt{a^2(t-kh)}}$, and in the integral I_2 $\alpha = \frac{\xi + x}{2\sqrt{a^2(t-kh)}}$. Then

$$I_1 = 2\int_{-z}^{\infty} e^{-\alpha^2}d\alpha,\ I_2 = 2\int_{z}^{\infty} e^{-\alpha^2}d\alpha,$$

where $z = \frac{x}{2\sqrt{t-kh}}$ and we get an estimate

$$|u_1(x,t)| \le M\mathrm{erf}\,(-\frac{x}{2\sqrt{t-kh}}),, \qquad (18)$$

$x > 0$, $kh < t < (k+1)h$.

So, the following theorem is proved.

Theorem 1. If there exists a number $M > 0$ such that the initial function $\varphi(x,kh)$ is bounded when $x > 0$, $h > 0$, $k \in$ N, $|\varphi(x,kh| \le M$, then the function $u_1(x, t)$ (16) when $x > 0$, $kh < t < (k+1)h$ is also bounded and the estimate (18) is true for it.

If $\varphi(\xi,kh) = \varphi_0$, where φ_0 is a number, then

$$u_1(x,t) = \varphi_0 \mathrm{erf}\left(\frac{x}{2\sqrt{t-kh}}\right),$$

$x > 0, kh < t < (k+1)h,$ $\mathrm{erf}\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\{-\xi^2\}d\xi$ is the error function.

By direct verification, it is possible to make sure that the Green's function (17) satisfies the homogeneous heat conduction equation (item 2 of the definition). When formally differentiating the function (16) under the sign of the integral, we obtain expressions

$$\frac{1}{(a^2(t-kh))^r}\int_0^\infty \varphi(\xi,kh)\,|x \pm y|^m\, e^{-\frac{(x\pm y)^2}{4a^2(t-kh)}}dy,$$

$x > 0$, $kh < t < (k+1)h$, where integrable functions are majored by an expression of the form $M\,|\xi|^m e^{-\xi^2}$ that is integrable on the entire numerical axis. This ensures uniform convergence of the integrals obtained after differentiation under the sign of the integral. Then the Poisson integral (16) is a continuous function, differentiable of arbitrary order with respect to *x* and *t* when $x > 0$, $kh < t < (k+1)h$, $k \in$ N, bounded with a bounded initial function, satisfying the homogeneous heat conduction equation (8) , since the Green's function (17g) satisfies equation (8). The implementation of the initial condition (9) and the boundary condition (10) is carried out similarly as in [4, 8]. Let us prove the uniqueness theorem of the solution to problem 1.

Theorem 2 (the uniqueness for an infinite straight line).

Let there be a number $M > 0$ such that in the domain $x \ge 0$ and $kh \le t \le (k+1)h, k \in N$ the functions $u_1(x,\ t)$ and $u_2(x,\ t)$ are bounded, that is $|u_i(x,t)| < M,\ \ i = 1,2$, satisfy the equation (8) and condition

$u_1(x,kh) = u_2(x,kh), x \le 0, k \in N,$

then

$u_1(x,t) = u_2(x,t), x \ge 0, kh \le t \le (k+1)h$.

Proof. Consider the function

$$v(x,t) = u_1(x,t) - u_2(x,t),$$

which is continuous, equation (8), bounded by

$$|v(x,t)| \le |u_1(x,t)| + |u_2(x,t)| < 2M,$$
$$x \ge 0, kh \le t \le (k+1)h, v(x,kh) = 0.$$

Consider the domain $0 \le x \le L, kh \le t \le (k+1)h$, where *L* is a real number and a function

$$V(x,t) = \frac{4M}{L^2}\Big(\frac{x^2}{2} + a^2(t-kh)\Big),$$

for which

$$\frac{\partial V}{\partial t} = \frac{4Ma^2}{L^2},\ \frac{\partial V}{\partial x} = \frac{4Mx}{L^2}, \frac{\partial^2 V}{\partial x^2} = \frac{4M}{L^2}$$

and which satisfies the thermal conductivity equation (8), as well as

$$V(x,kh) \ge v(x,kh) = 0$$

$$V(\pm L,t) \ge 2M \ge | v(\pm L,t) |, \tag{19}$$

For each limited region $0 \le x \le L$, $kh \le t \le (k+1)h, k \in N$, the principle of the maximum value is true [4, p. 194]. From Corollary 2 [4, p. 198] for the functions $\underline{u} = -V(x,t)$, $u = v(x,t)$, $\bar{u} = V(x,t)$, taking into account (19), we obtain that

$$-\frac{4M}{L^2}\Big(\frac{x^2}{2} + a^2(t-kh)\Big) \le v(x,t) \le$$

$$\le \frac{4M}{L^2}\Big(\frac{x^2}{2} + a^2(t-kh)\Big).$$

We fix (x, t) and use the fact that L is arbitrary and can be increased indefinitely. Passing to the limit at $L \to \infty$, we obtain that $v(x,t) \equiv 0$ for $x \ge 0$, $kh \le t \le (k+1)h$

Theorem 2 is proved.

Therefore, the following theorem is true.

Theorem 3. If $|\varphi(x,h)| \le M$, $x \ge 0$, $M > 0$, $h > 0$, then the solution of problem (8), (9), (10) exists, is unique and is determined by formula (16).

2.3. Solving the problems 2 and 3

It is necessary to solve equation (8) when the zero initial condition (11) and the general boundary condition (12) are met. First, let's solve the auxiliary problem of cooling a heated rod, at the boundary of which a constant zero temperature is maintained. Then, for equation (8), the Cauchy condition and the boundary condition are given as follows:

$$V_1(x,t_0) = T,\ v_1(0,\ t) = 0,\ x > 0,\ t > h.$$

Then, according to formula (16), we get that

$$\bar{v} = T\ \mathrm{erf}\Big(\frac{x}{2\sqrt{a^2(t-t_0)}}\Big),\ x \ge 0,\ t > t_0, \tag{20}$$

Let $\mu(t) = \mu_0 \equiv const$ in condition (12). Then, according to (20), the function

$$\bar{v} = \mu\ \mathrm{erf}\Big(\frac{x}{2\sqrt{a^2(t-t_0)}}\Big),\ x \ge 0,\ t > t_0,$$

is a solution of problem (8), (11), (12). Then the function

$$v(x,t) = \mu_0 - \bar{v}(x,t) = \mu_0\left[1\text{- erf}\Big(\frac{x}{2\sqrt{a^2(t-t_0)}}\Big)\right],$$

$$x > 0,\ t > 0. \tag{21}$$

We denote the expression in parentheses of formula (21) by $U(x, t - t_0)$, which makes sense when $t > t_0$. If for $t < t_0$ the value of this function is extended by zero, then this definition is consistent with the zero value of the function at $t = t_0$. The limit value of this function at $x = 0$ is a step function equal to zero at $t < t_0$ and equal to 1 at $t > t_0$. The constructed function is often found in applications and is an auxiliary link in constructing the solution to problem 2.

The second auxiliary task is to find a solution of the equation (8) under the following conditions:

$$v(x,t_0) = 0, x \ge 0,$$

$$v(0,t) \equiv \mu(t) = \begin{Bmatrix} \mu_0, t_0 < t < t_1, \\ 0, t > t_1 \end{Bmatrix}.$$

It is directly verified that

$$V(x,t) = \mu_0[U(x,t-t_0) - U(x,t-t_1)],$$

$$x \ge 0, t > t_0.$$

If

$$\mu(t) = \begin{cases} \mu_0, t_0 < t \le t_1, \\ \mu_1, t_1 < t \le t_2, \\ \dots\dots\dots \\ \mu_{n-1}, t_{n-2} < t \le t_{n-1}, \\ \mu_{n-1}, t_{n-1} < t \le t_n, \end{cases},$$

and then the solution of the corresponding problem can be written in the form

$$u(x,t) = \sum_{i=0}^{n-2} \mu_i [U(x,t-t_i) - U(x,t-t_{n+1})] +$$

$$+\ \mu_{n-1} U(x,t-t_{n-1})$$

Using the theorem on finite increments, we get

$$u(x,t) = \sum_{i=0}^{n-2} \mu_i \frac{\partial}{\partial t} U(x,t-\tau)\Big|_{\tau=\tau_i} +$$

$$+\ \mu_{n-1} U(x,t-t_n), \tag{22}$$

where $x \ge 0$, $t_i \le \tau_i \le t_{i+1}$

The approximate solution of problem 2 can be obtained by formula (22), if replace the function $\mu(t)$ with a piecewise-constant function.

Heading to the limit when the interval of constancy of the auxiliary function decreases, we obtain that the limit of the sum (22) will take the form

$$\int_0^t \frac{\partial U}{\partial t}(x,t-\tau)\mu(\tau)d\tau ,$$

because when $x \geq 0,$ we have

$$\lim_{t-t_{n-1}\to 0} \mu_{n-1}U(x,t-t_{n-1}) = 0 .$$

If we consider that

$$\frac{\partial U}{\partial t}(x,t) = -2a^2 \frac{\partial G}{\partial x}(x,0,t) = 2a \left.\frac{\partial G}{\partial \xi}\right|_{\xi=0}$$

then we will get the final result

$$u_2(x,t) = \frac{a^2}{2\sqrt{\pi}} \int_{kh}^{t} \frac{x}{[a^2(t-\tau)]^{3/2}} \times$$
$$\times \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\mu(t)d\mu , \qquad (23)$$

$x > 0,$ $kh \leq t \leq (k+1)h$.

The solution of problem 3 using the Green's function (17) can be written in the form of a Poisson integral

$$u_3(t,x) = \int_{kh}^{t} d\tau \int_0^{\infty} f(y,\tau)G_1(x,y,t-kh)dy \quad (24)$$

$x > 0$, $kh \leq t \leq (k+1)h$, $k \in$ N, for the existence of which the function $f(x,t)$ must be such that the improper integral in formula (24) coincides.

So, the following theorem is proved.

Theorem 4. The solution of problem (8), (11), (12) is determined by formula (24). The solution of problem (4), (5), (6) is determined by formula (7), where the terms u_1, u_2, u_3 are the solutions of problems 1, 2, and 3, respectively.

CONCLUSION

In this paper was formulated the first boundary value problem for the heat conduction equation containing a nonlinear term dependent on the searched function with a deviation of the argument for the first time. For such equations, the initial condition is set on a certain interval. Physical and technical reasons for lateness can be transport delays, delays in information transmission, delays in decision-making, etc. The most natural are delays when modeling objects in ecology, medicine, population dynamics, etc. Features of the dynamics of vehicles in different environments (water, land, air) can also be taken into account by introducing a delay. Other physical and technical interpretations are also possible. The study of the molecular distribution of heat energy in various substances (solid bodies, liquids, etc.) leads to heat conduction equations. The Green's function of the first boundary value problem is constructed for the nonlinear equation of heat conduction with a deviation of the argument, its properties are studied, and the formula for the solution is established.

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