*Journal of Engineering Science* *Vol. XXVII, no. 2 (2020), pp. 37 - 43*
*Fascicle* *Industrial Engineering* *ISSN 2587-3474*
*Topic* *Applied Engineering Sciences and Management* *eISSN 2587-3482*

**DOI: 10.5281/zenodo.3784305**
**CZU 519.857**

# STOCHASTIC OPTIMAL CONTROL OF A TWO-DIMENSIONAL DYNAMICAL SYSTEM

Mario Lefebvre*, ORCID ID: 0000-0002-9451-543X

*Polytechnique Montréal, 2500, chemin de Polytechnique, Montréal, Canada*
**mlefebvre@polymtl.ca*

Received: 03. 22.2020
Accepted: 05. 04.2020

**Abstract.** In this paper, we considered the problem of optimally controlling a two-dimensional dynamical system until it reaches either of two boundaries. We consider a controlled dynamical system $(X(t), Y(t))$ which is a generalization of the classic two-dimensional Kermack-McKendrick model for the spread of epidemics. Moreover, the system is subject to random jumps of fixed size according to a Poisson process. The system is controlled until the sum $X(t) + Y(t)$ is equal to either 0 or *d* (> 0) for the first time. Particular problems are solved explicitly.

**Keywords:** *Dynamic programming, error function, first-passage time, random jumps, Poisson process.*

## 1. Introduction

Let $X(t)$ denote the number (or percentage) of individuals in a certain population who are susceptible to a virus, and let $Y(t)$ be the number (or percentage) of infected carriers. The two-dimensional model proposed by Kermack and McKendrick [1] for the spread of epidemics is the following:

$$\dot{X}(t) = -k_1 X(t) Y(t), \tag{1}$$

$$\dot{Y}(t) = k_1 X(t) Y(t) - k_2 Y(t), \tag{2}$$

where $k_1$ and $k_2$ are positive constants. There is also a three-dimensional version of the model in which the variable $Z(t)$ represents the number (or percentage) of individuals who are either recovered or removed from the population and is such that

$$\dot{Z}(t) = k_2 Y(t), \tag{3}$$

This type of model is known as a SIR model in epidemiology, for Susceptible, Infected and Recovered (or Removed). Depending on the application, an individual can be removed from the population because he/she is cured and immune, or is dead.

In this paper, we first generalize the system (1), (2) and we introduce a control variable:

$$\dot{X}(t) = -k_1 X(t) Y(t), \tag{4}$$

$$\dot{Y}(t) = k_1 X(t)Y(t) + f[X(t), Y(t)] + b[X(t), Y(t)]u(t). \quad (5)$$

Then, we assume that there are random jumps of fixed size $\epsilon$ (> 0) that occur according to a Poisson process $N(t)$ with rate $\lambda$, so that Eq. (5) becomes

$$\dot{Y}(t) = k_1 X(t)Y(t) + f[X(t), Y(t)] + b[X(t), Y(t)]u(t) + \epsilon \dot{N}(t). \quad (6)$$

Let $T(x, y)$ be the random variable defined by

$$T(x, y) = \inf\{t > 0: X(t) + Y(t) = 0 \text{ or } d \mid X(0) = x, Y(0) = y\}, \quad (7)$$

where 0 < *x* + *y* < *d*. We are looking for the control $u(t)$ that minimizes the expected value of the cost function

$$J(x, y) = \int_0^T \frac{1}{2} q[X(t), Y(t)]u^2(t)\,\mathrm{d}t + K[X(T), Y(T)], \quad (8)$$

in which $q[X(t), Y(t)]$ is a positive function and we choose the following final cost:

$$K[X(T), Y(T)] = \begin{cases} K_1 \text{ if } X(T) + Y(T) = 0, \\ K_2 \text{ if } X(T) + Y(T) = d, \end{cases} \quad (9)$$

where $0 \leq K_1 < K_2$. Therefore, the optimizer wants to end the epidemic as soon as possible, while taking the quadratic control costs into account. If the sum $X(T) + Y(T)$ reaches the value *d*, it is too expensive to bring the sum to zero by using the control variable $u(t)$.

The problem set up above is known as an *LQG homing* problem; see Whittle [2] and [3]. However, here we assume that there are random jumps instead of a Gaussian white noise; see Lefebvre [4]. There could be both a Gaussian white noise and jumps according to a Poisson process, so that *Y*(*t*) would be a controlled jump-diffusion process. This type of problem was considered by Lefebvre [5] and by Lefebvre and Moutassim [6].

To solve our problem, we will make use of dynamic programming. In the next section, we will derive the equation satisfied by the value function $F(x, y)$, which is defined as follows:

$$F(x, y) = \inf_{u(t), 0 \leq t \leq T} E[J(x, y)]. \quad (10)$$

Then, in Section 3, particular problems will be considered and solved explicitly. Finally, we will end this paper with some concluding remarks.

## 2. Dynamic programming

With the help of Bellman's principle of optimality, we can write that

$$F(x, y) = \inf_{u(t), 0 \leq t \leq \Delta t} E\left\{\int_0^{\Delta t} \frac{1}{2} q[X(t), Y(t)]u^2(t)\,\mathrm{d}t + F\big(X(\Delta t), Y(\Delta t)\big)\right\}, \quad (11)$$

where

$$F\big(X(\Delta t), Y(\Delta t)\big) = F\big(x - k_1 xy\Delta t, y + [k_1 xy + f + bu(0)]\Delta t + \epsilon N(\Delta t)\big) + o(\Delta t) \quad (12)$$

We have

$$\int_0^{\Delta t} \frac{1}{2} q[X(t), Y(t)]u^2(t)\,\mathrm{d}t = \frac{1}{2} q(x, y)\, u^2(0)\Delta t + o(\Delta t) \quad (13)$$

and, from Taylor's formula,

$$\begin{aligned} &F\big(x - k_1 xy\Delta t, y + [k_1 xy + f + bu(0)]\Delta t + \epsilon N(\Delta t)\big) \\ &= F(x, y) - k_1 xy\Delta t F_x(x, y) + \{[k_1 xy + f + bu(0)]\Delta t + \epsilon N(\Delta t)\}F_y(x, y) + o(\Delta t) \end{aligned} \quad (14)$$

Moreover, for a Poisson process, we can write that

$$P[N(\Delta t) = 0] = e^{-\lambda \Delta t} = 1 - \lambda \Delta t + o(\Delta t) \tag{15}$$

And

$$P[N(\Delta t) = 1] = \lambda \Delta t e^{-\lambda \Delta t} = \lambda \Delta t + o(\Delta t). \tag{16}$$

It follows that

$$0 = \inf_{u(t), 0 \le t \le \Delta t} \Big\{ \Delta t \Big[ \tfrac{1}{2} q u^2 - k_1 x y F_x + [k_1 x y + f + b u] F_y + \lambda [F(x, y + \epsilon) - F(x, y)] \Big] + o(\Delta t) \Big\} \tag{17}$$

where all the functions are evaluated at $t = 0$. Dividing each side of Eq. (17) by $\Delta t$, and letting $\Delta t$ decrease to zero, we obtain the *dynamic programming equation*

$$0 = \inf_u \Big\{ \tfrac{1}{2} q u^2 - k_1 x y F_x + (k_1 x y + f + b u) F_y + \lambda [F(x, y + \epsilon) - F(x, y)] \Big\}. \tag{18}$$

Hence, the optimal control $u^*$ can be expressed in terms of the value function as follows:

$$u^* = -\frac{b}{q} F_y . \tag{19}$$

Therefore, to obtain the optimal control, we must solve the non-linear partial differential-difference equation

$$-\frac{b^2}{2q} \left(F_y\right)^2 - k_1 x y F_x + (k_1 x y + f) F_y + \lambda [F(x, y + \epsilon) - F(x, y)] = 0 . \tag{20}$$

This equation is valid for $0 < x + y < d$. Moreover, we have the boundary conditions

$$F(x, y) = K_1 \text{ if } x + y = 0 \quad \text{and} \quad F(x, y) = K_2 \text{ if } x + y = d. \tag{21}$$

Based on the above conditions, to solve Eq. (20) we will look for solutions of the form

$$F(x, y) = H(w), \tag{22}$$

where $w := x + y$. This a particular case of the *method of similarity solutions*, and $w$ is called the *similarity variable*. For the method to apply, we must be able to express Eq. (20) in terms of $w$, as well as the boundary conditions in Eq. (21). These conditions become

$$H(0) = K_1 \text{ and } H(d) = K_2 . \tag{23}$$

Furthermore, we have

$$F(x, y + \epsilon) - F(x, y) = \epsilon F_y + \frac{\epsilon^2}{2} F_{yy} + o(\epsilon^2). \tag{24}$$

Therefore, if $\epsilon$ is small, we can write that

$$-\frac{b^2}{2q} \left(F_y\right)^2 - k_1 x y F_x + (k_1 x y + f) F_y + \lambda \left( \epsilon F_y + \frac{\epsilon^2}{2} F_{yy} \right) \cong 0 . \tag{25}$$

This equation reduces to the ordinary differential equation

$$-\frac{b^2}{2q} (H')^2 + f H' + \lambda \left( \epsilon H' + \frac{\epsilon^2}{2} H'' \right) \cong 0 . \tag{26}$$

We can now state the following proposition.

**Proposition 1.** If the ratio $b^2/q$ and the function $f$ can be expressed in terms of the similarity variable $w$, and if the jump size $\epsilon$ is small, then the optimal control $u^*$ can be obtained (approximately) from the solution of Eq. (26), subject to the boundary conditions in Eq. (23).

In the next section, particular problems will be considered and solved explicitly.

### 3. Particular problems

Assume that the ratio $b^2/q$ is a constant:

$$\frac{b^2}{2q} \equiv \kappa \ (> 0). \tag{27}$$

First, we consider the case when $f(x, y) \equiv \gamma$. Let $\gamma + \lambda\epsilon = \alpha$, and $\lambda\epsilon^2/2 = \beta$. Then, we must solve the second-order non-linear ordinary differential equation

$$-\kappa(H')^2 + \alpha H' + \beta H'' \cong 0\,. \tag{28}$$

Notice that this equation is a *Riccati equation* for $G(w) := H'(w)$. The solution of Eq. (28) that satisfies the boundary conditions in Eq. (23) is

$$H(w) = -\frac{\beta}{\kappa}\ln\left[\frac{\exp\left(-\frac{\kappa K_2+\alpha w}{\beta}\right)-\exp\left(-\frac{\kappa K_1+\alpha w}{\beta}\right)+\exp\left(-\frac{\kappa K_1+\alpha d}{\beta}\right)-\exp\left(-\frac{\kappa K_2}{\beta}\right)}{\exp\left(-\frac{\alpha d}{\beta}\right)-1}\right]. \tag{29}$$

From this solution, we can compute at once the optimal control given in Eq. (19) for any choice of the functions $b$ and $q$ such that Eq. (27) is satisfied.

To illustrate the results that we obtained, let us consider the particular case when $\kappa$ = 2, $\gamma$ = −1, $\lambda$ = 10, $d$ = 2, $K_1$ = 1 and $K_2$ = 2. The function $H(w)$ is shown in Figure 1 for various values of the parameter $\epsilon$. The corresponding optimal controls (in terms of $w$) in the case when $b \equiv 2$ and $q \equiv 1$ (so that $\kappa$ = 2, as required), are given by $u^* = -2H'(w)$.

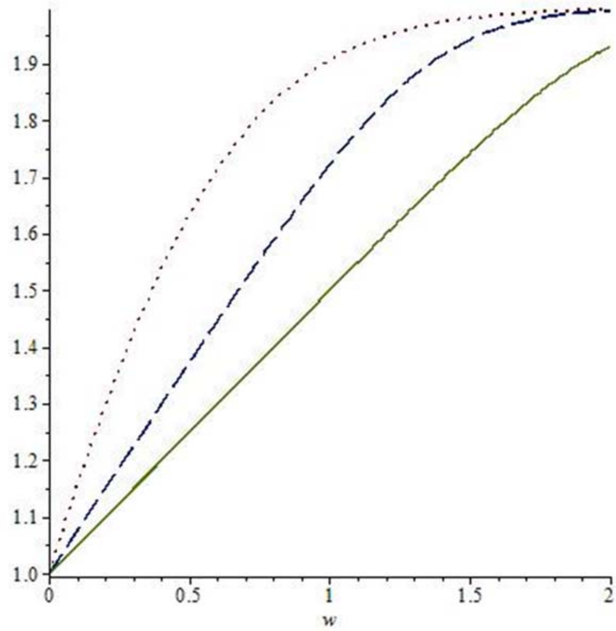


**Figure 1.** Function $H(w)$ when $f \equiv -1$ and (from left to right) $\epsilon$ = 0.5, 0.25 and 0.2.

When we substitute the approximate solution $H(w)$ into Eq. (20), we obtain the curves presented in Figure 2. We see that when $\epsilon$ = 0.25 (the dashed curve), the error (that is, the difference between zero and the value of the equation for $w$ in the interval $[0, 2]$) is already quite small. With $\epsilon$ = 0.2 (the solid curve), the error is almost equal to zero, except near $w$ = 1.5. Notice that our approximate solution should only be used when $w + \epsilon \leq 2$.

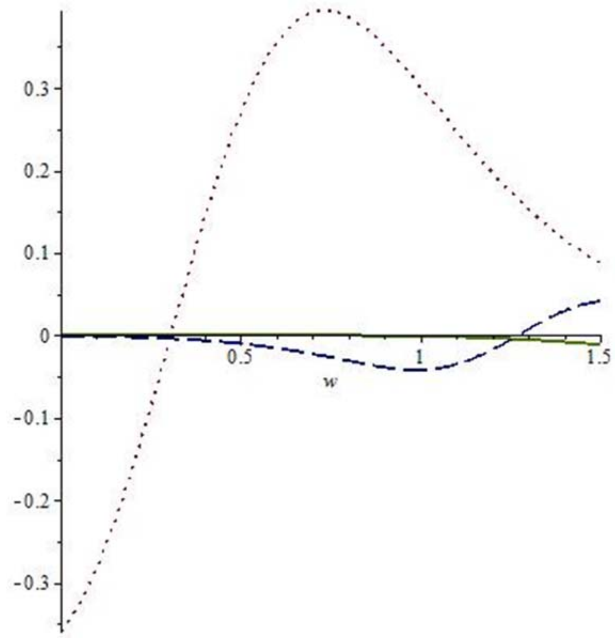


**Figure 2.** Error obtained with $H(w)$ when $f \equiv -1$ and $\epsilon$ = 0.5, 0.25 and 0.2.

In the second particular case that we consider, we assume now that $f(x, y) = \gamma(x + y)$ and that Eq. (27) still holds. According to Proposition 1, we can again make use of the method of similarity solutions to solve our problem. With the same values for the various parameters that were used above, we must now solve the differential equation

$$-2\,(H')^2 - wH' + 10\left(\epsilon H' + \frac{\epsilon^2}{2} H''\right) \cong 0\,. \tag{30}$$

We can obtain the general solution of this equation. It is expressed in terms of the *error function* erf:

$$H(w) = -\frac{5\epsilon^2}{2} \ln\left[\frac{e^{-\frac{4}{5\epsilon^2}}\left(\mathrm{erf}\left(i\sqrt{10}\right) - E_1\right) + e^{-\frac{2}{5\epsilon^2}}(E_1 - E_2)}{\mathrm{erf}\left(i\sqrt{10}\right) - E_2}\right], \tag{31}$$

where

$$E_1 := \mathrm{erf}\left(\frac{i(10\epsilon - w)}{\sqrt{10}\epsilon}\right) \text{ and } E_2 := \mathrm{erf}\left(\frac{i\sqrt{10}(5\epsilon - 1)}{5\epsilon}\right). \tag{32}$$

The solution that satisfies the boundary conditions $H(0) = 1$ and $H(2) = 2$ is shown in Figure 3, for $\epsilon$ = 0.5, 0.25 and 0.2.

Finally, we present the error obtained when using the approximate solution $H(w)$ in Figure 4. Again, we see that the error is very small when $\epsilon$ = 0.25 and 0.2.

Now, the cost function $J(x, y)$ defined in Eq. (8) can be generalized as follows:

$$C(x, y) = \int_0^T \left\{\frac{1}{2} q[X(t), Y(t)]u^2(t) + \theta\right\} \mathrm{d}t + K[X(T), Y(T)], \tag{33}$$

where $\theta$ is a real parameter. When $\theta$ is positive (respectively negative) and $K[X(T), Y(T)] = 0$, the optimizer tries to reach either boundary as soon (respectively as late) as possible.

If $\theta = 1$ and we use the same parameters as in the first particular case considered above, we must solve the non-homogeneous differential equation

$$1 - 2(H')^2 + (-1 + 10\epsilon)H' + 5\epsilon^2 H'' \cong 0\,, \tag{34}$$

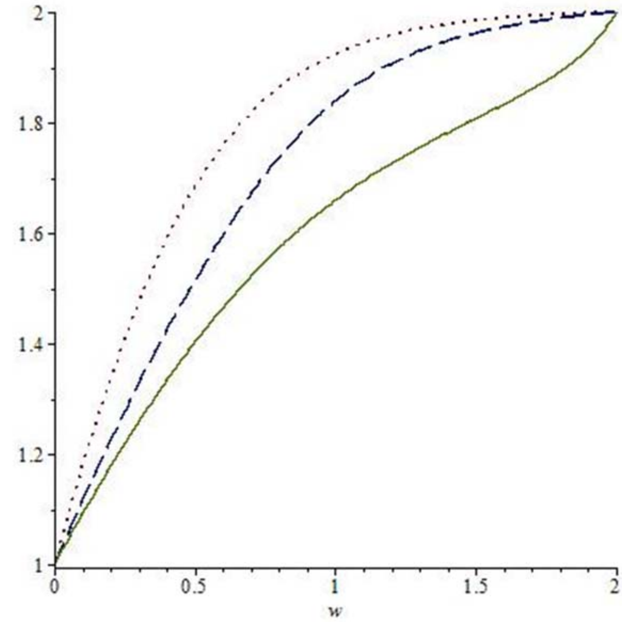


**Figure 3.** Function $H(w)$ when $f = -(x+y)$ and (from left to right) $\epsilon$ = 0.5, 0.25 and 0.2.

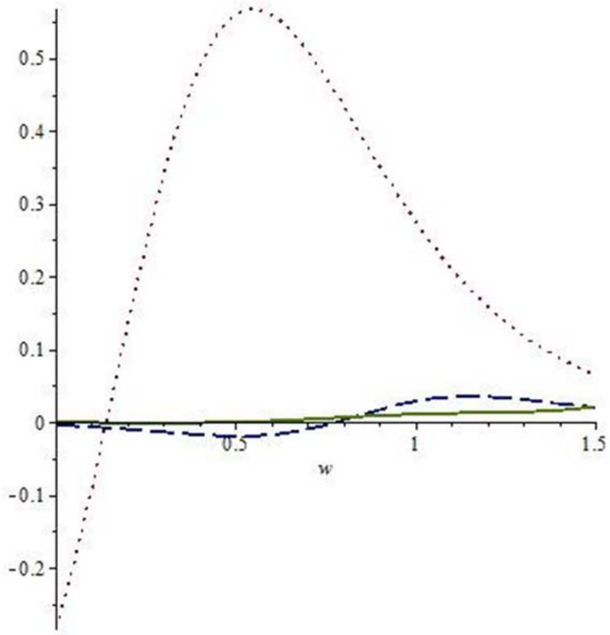


**Figure 4.** Error obtained with $H(w)$ when $f = -(x+y)$ and $\epsilon$ = 0.5, 0.25 and 0.2.

subject to the boundary conditions $H(0) = 1$ and $H(2) = 2$. We find that

$$H(w) = 1 + \frac{5}{2}w\epsilon + \frac{1}{4}w(\delta - 1) - \frac{5\epsilon^2}{2}\ln\left[\frac{\exp\left(\frac{\delta(w+1)+10\epsilon+1}{5\epsilon^2}\right) - \exp\left(\frac{\delta+10\epsilon+1}{5\epsilon^2}\right) - \exp\left(\frac{w\delta+4}{5\epsilon^2}\right) + \exp\left(\frac{2\,(2+\delta)}{5\epsilon^2}\right)}{\exp\left(\frac{2\,(2+\delta)}{5\epsilon^2}\right) - \exp\left(\frac{4}{5\epsilon^2}\right)}\right] \quad (35)$$

where

$$\delta := \sqrt{100\epsilon^2 - 20\epsilon + 9}\,. \quad (36)$$

The function $H(w)$ is shown in Figure 5 for $\epsilon$ = 0.5, 0.25 and 0.2.

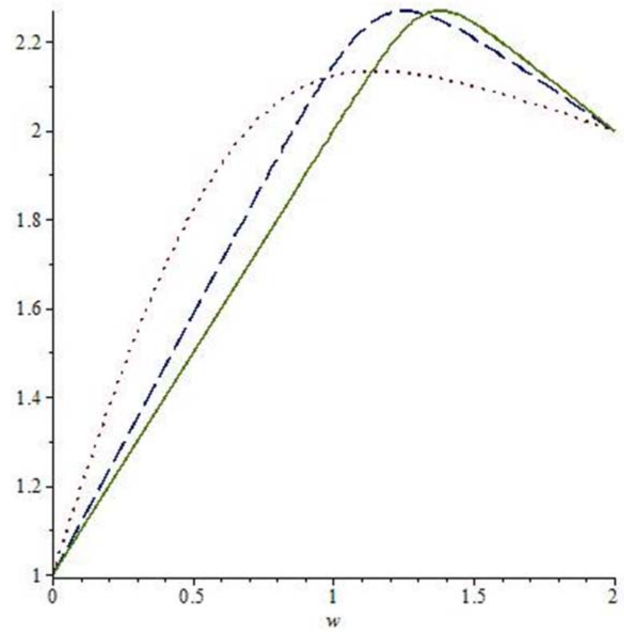


**Figure 5.** Function $H(w)$ when $f \equiv -1$ and $\theta = 1$ for (from left to right) $\epsilon = 0.5$, 0.25 and 0.2.

## 4. Concluding remarks

We considered the problem of optimally controlling a two-dimensional dynamical system until it reaches either of two boundaries. The system was subject to random jumps of fixed positive size, according to a Poisson process. As a generalization of this work, we could assume that the jumps are of random size and could be positive or negative.

We were able to obtain explicit approximate solutions to particular problems by making use of the method of similarity solutions. We saw that the error obtained by using these approximate solutions was very small when the jump size is also small, as expected. When this technique does not apply, we could at least try to solve the appropriate differential equations numerically.

Finally, this type of optimal control problem, when the final time is a random variable, could be considered for other important dynamical systems, for instance the classic predator-prey model of Volterra and Lotka [7].

**Acknowledgments.** This research was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

**References**

1. Kermack W.O., McKendrick A.G. A contribution to the mathematical theory of epidemics. In: *Proceedings of the Royal Society of London, Series A*, 1927, 115, pp. 700-721.
2. Whittle P. *Optimization over Time*, Vol. I. Chichester: Wiley, 1982.
3. Whittle P. *Risk-sensitive Optimal Control*. Chichester: Wiley, 1990.
4. Lefebvre M. Optimal control of a stochastic system related to the Kermack-McKendrick model. (Submitted for publication)
5. Lefebvre M. LQG homing for jump-diffusion processes. In: *ROMAI Journal*, 2014, 10 (2), pp. 1-6.
6. Lefebvre M., Moutassim A. Exact solutions to the homing problem for a Wiener process with jumps. In: *Optimization*. (To appear)
7. Qian He Uwe C. T¨Auber and R. K. P. ZIA. On the relationship between cyclic and hierarchical three-species predator-prey systems and the two-species Lotka–Volterra model. 2018, pp.13. https://arxiv.org/pdf/1111.1674.pdf.