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On the left coquotient with respect to meet for pretorsions in modules

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Abstract

In [2] the operation of *left coquotient with respect to meet* for preradicals of R-Mod is defined. In the present short notice the particular case of *pretorsions* of R-Mod is considered. We prove that for pretorsions the studied operation coincides with the operation (called *right residual*) introduced and investigated by J.S.Golan ([1]) in the terms of preradical filters of R. For that it is necessary to show the concordance of the studied operation with the transition $r \rightsquigarrow \mathcal{E}_r$ from pretorsions of R-Mod to the preradical filters of the ring R.

Keywords: module, pretorsion, filter, left coquotient.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. By definition a pretorsion is a hereditary preradical. We denote by $\mathbb{PT}$ the set of all pretorsions of the category R-Mod. It is well known the description of pretorsions by preradical filters.

Definition. *The set of left ideals $\mathcal{E} \subseteq \mathbb{L}(_RR)$ is called* preradical filter *(left linear topology) if it satisfies the following conditions:*

(a_1) *If $I \in \mathcal{E}$ and $a \in R$, then $(I : a) = \{x \in R \mid xa \in I\} \in \mathcal{E}$;*

(a_2) *If $I \in \mathcal{E}$ and $I \subseteq J$, $J \in \mathbb{L}(_RR)$, then $J \in \mathcal{E}$;*

(a_3) *If $I, J \in \mathcal{E}$, then $I \cap J \in \mathcal{E}$.*

There exists a monotone bijection between the pretorsions of R-Mod and preradical filters of $\mathbb{L}(_RR)$ defined by the mappings:

$$r \rightsquigarrow \mathcal{E}_r, \quad \mathcal{E}_r = \{I \in \mathbb{L}(_RR) \mid r(R/I) = R/I\};$$

$$\mathcal{E} \rightsquigarrow r_{\mathcal{E}}, \quad r_{\mathcal{E}}(M) = \{m \in M \mid (0 : m) \in \mathcal{E}\}$$ ([3],[4]).

We denote by $\mathbb{PF}$ the set of all preradical filters of the lattice $\mathbb{L}(_R R)$ of left ideals of R. The sets $\mathbb{PT}$ and $\mathbb{PF}$ can be considered as complete lattices and the mappings indicated above determine an isomorphism of these lattices: $\mathbb{PT} \cong \mathbb{PF}$.

We mention that in the lattice $\mathbb{PT}$ the product of two pretorsions $r \cdot s$ coincides with their meet $r \wedge s$.

In $\mathbb{PT}(\wedge, \vee)$ we also have the operation $r \# s$ defined by the rule $[(r \# s)(M)]/s(M) = r(M/s(M))$, $M \in R$-Mod and $r \# s$ is called the *coproduct* of pretorsions r and s.

In a similar way is introduced in $\mathbb{PF}$ the notion of coproduct:
$\mathcal{E}_r \# \mathcal{E}_s = \{I \in \mathbb{L}(_R R) \,|\, \exists\, H \in \mathcal{E}_r,\ I \subseteq H$ such that $(I : a) \in \mathcal{E}_s, \forall a \in H\}$.

So we have the isomorphic lattices $\mathbb{PT}(\wedge, \vee, \#)$ and $\mathbb{PF}(\bigwedge, \bigvee, \#)$ with the following properties:

$$\mathcal{E}_{\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha} = \bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}; \quad \mathcal{E}_{\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha} = \bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}.$$

Now we remind some notions and results of the monograph [1], where the pretorsions of R-Mod are investigated by the point of view of the associated preradical filters. In [1] $\mathbb{PF}$ is denoted by $R - fil$ and the operation of *multiplication* in $R - fil$ is defined by the rule:
$KK' = \{I \in \mathbb{L}(_R R) \,|\, \exists\, H \in K'$, such that $I \subseteq H$ and $(I : a) \in K, \forall a \in H\}$, where $K, K' \in R - fil$.

It is easy to see that in our notations for every $r, s \in \mathbb{PT}$ we have $\mathcal{E}_s \mathcal{E}_r = \mathcal{E}_r \# \mathcal{E}_s$. All properties of the operation of multiplication easily can be translated in the language of coproduct, in particular associativity and distributivity:

$$\mathcal{E}_1 \# (\mathcal{E}_2 \# \mathcal{E}_3) = (\mathcal{E}_1 \# \mathcal{E}_2) \# \mathcal{E}_3; \quad (\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_{r_\alpha}) \# \mathcal{E} = \bigwedge_{\alpha\in\mathfrak{A}} (\mathcal{E}_{r_\alpha} \# \mathcal{E}).$$

Using the product KK' of preradical filters, in [1] is defined *right residual* $K'^{-1}K$ of K by K' as the unique minimal preradical filter K'' in $R - fil$ satisfing $K'K'' \supseteq K$. By the distributivity such a filter always exists and is equal to $\bigcap\{K'' \,|\, K'K'' \supseteq K\}$. In the book [1] a series of properties of this operation is exposed.

Translating in our notations and making the necessary changes (multiplication versus coproduct) we obtain the following statements.

Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{PF}$. *Left coquotient with respect to meet* of $\mathcal{E}_1$ by

$\mathcal{E}_2$ is called the minimal preradical filter $\mathcal{E}$ such that $\mathcal{E} \# \mathcal{E}_2 \supseteq \mathcal{E}_1$ or $\bigwedge\{\mathcal{E} \in \mathbb{PF} \,|\, \mathcal{E} \# \mathcal{E}_2 \supseteq \mathcal{E}_1\}$. The distributivity ensures the existence of this coquotient, denoted by $\mathcal{E}_1 \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_2$ ([2]).

Now we will show that this preradical filter coincides with the preradical filter of the pretorsion $r_{\mathcal{E}_1} {}^{\wedge}\!/_{\#}\, r_{\mathcal{E}_2}$.

Lemma. *If* r, $s \in \mathbb{PT}$, *then* $\mathcal{E}_{r \# s} = \mathcal{E}_r \# \mathcal{E}_s$.

Proof. Firstly we specify the expresions of pretorsions determined by indicated preradical filters, using that $r(M) = \{m \in M \mid (0:m) \in \mathcal{E}_r\}$ for every $r \in \mathbb{PT}$ and $M \in R$-Mod.

The preradical filter $\mathcal{E}_{r \# s}$ is determined by the pretorsion $r \# s$ and $(r \# s)(M) = \{m \in M \,|\, (m + s(M)) \in r(M/s(M))\}$. But $r(M/s(M)) = \{x + s(M) \,|\, x \in M$ and $(0 : (x + s(M))) \in \mathcal{E}_r\} = \{x + s(M) \,|\, x \in M$ and $(s(M) : x) \in \mathcal{E}_r\}$, so we have $(r \# s)(M) = \{m \in M \,|\, (s(M) : m) \in \mathcal{E}_r\}$.

We denote by t the pretorsion of R-Mod defined by $\mathcal{E}_r \# \mathcal{E}_s$, so for every $M \in R$-Mod we have $t(M) = \{m \in M \,|\, (0:m) \in \mathcal{E}_r \# \mathcal{E}_s\} = \{m \in M \,|\, \exists H \in \mathcal{E}_r,\ (0:m) \subseteq H$ such that $((0:m):a) \in \mathcal{E}_s, \forall a \in H\} = \{m \in M \,|\, \exists H \in \mathcal{E}_r,\ (0:m) \subseteq H$ such that $(0:am) \in \mathcal{E}_s$, $\forall a \in H\}$.

Now we verify the equality of lemma.

($\subseteq$) It is sufficient to show that $r \# s \leq t$. For every $M \in R$-Mod if $m \in (r \# s)(M)$, then $H = (s(M) : m) \in \mathcal{E}_r$ and $(0:m) \subseteq (s(M) : m) = H$. So if $a \in H$, then $am \in s(M)$, i.e. $(0:am) \in \mathcal{E}_s$, which means that $m \in t(M)$. Therefore $(r \# s)(M) \subseteq t(M)$ for every $M \in R$-Mod, i.e. $r \# s \leq t$, which implies $\mathcal{E}_{r \# s} \subseteq \mathcal{E}_r \# \mathcal{E}_s$.

($\supseteq$) We verify that $t \leq r \# s$. Let $M \in R$-Mod and $m \in t(M)$. Then there exists $H \in \mathcal{E}_r$ such that $(0:m) \subseteq H$ and $(0:am) \in \mathcal{E}_s, \forall a \in H$. If $a \in H$, then $(0:am) \in \mathcal{E}_s$, so $am \in s(M)$, i.e. $a \in (s(M) : m)$, therefore $H \subseteq (s(M) : m)$. From the definition of preradical filter (condition (a_2)) since $H \in \mathcal{E}_r$ now we have $(s(M) : m) \in \mathcal{E}_r$, which means that $m \in (r \# s)(M)$. This proves that $t(M) \subseteq (r \# s)(M)$ for every $M \in R$-Mod, therefore $t \leq r \# s$ and so $\mathcal{E}_r \# \mathcal{E}_s \subseteq \mathcal{E}_{r \# s}$. □

Proposition. *For every pretorsions* r, $s \in \mathbb{PT}$ *we have:*

$$\mathcal{E}_{r \,{}^{\wedge}\!/_{\#}\, s} = \mathcal{E}_r \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_s .$$

Proof. ($\supseteq$) By definition $\mathcal{E}_r \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_s = \bigwedge\{\mathcal{E} \in \mathbb{PF} \,|\, \mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r\}$, i.e.

it is the least preradical filter with the property $\mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r$. From the Lemma $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s} \# \mathcal{E}_s = \mathcal{E}_{(r\,{}^{\wedge}\!/_{\#}\, s) \# s}$ and since $(r\,{}^{\wedge}\!/_{\#}\, s) \# s \geq r$ ([2]) we have $\mathcal{E}_{(r\,{}^{\wedge}\!/_{\#}\, s) \# s} \supseteq \mathcal{E}_r$, so $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s} \# \mathcal{E}_s \supseteq \mathcal{E}_r$. Therefore $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s}$ is one of preradical filter $\mathcal{E}$ and so $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s} \supseteq \bigwedge\{\mathcal{E} \in \mathbb{PF} \mid \mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r\}$, i.e. $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s} \supseteq \mathcal{E}_r \,{}^{\wedge}\!/_{\#}\, \mathcal{E}_s$.
$(\subseteq)$ Let $\mathcal{E}_t$ be preradical filter defined by pretorsion t with the property $\mathcal{E}_t \# \mathcal{E}_s \supseteq \mathcal{E}_r$. From the Lemma $\mathcal{E}_{t \# s} \supseteq \mathcal{E}_r$, therefore $t \# s \geq r$. Since $r\,{}^{\wedge}\!/_{\#}\, s$ is the least pretorsion h with the property $h \# s \geq r$ ([2]) it follows that $r\,{}^{\wedge}\!/_{\#}\, s \leq t$ i.e. $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s} \subseteq \mathcal{E}_t$. So $\mathcal{E}_{r\,{}^{\wedge}\!/_{\#}\, s}$ is the least between preradical filters $\mathcal{E}$ with the property $\mathcal{E} \# \mathcal{E}_s \supseteq \mathcal{E}_r$. □

As a conclusion we can affirm that all results of J.S.Golan [1] about the operation of right residual of preradical filters can be treated as a particular case of the operation of left coquotient with respect to meet, defined in [2] in general case of preradicals of $M \in R$-Mod.

References

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