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On the inverse operations in the class of preradicals of a module category, I

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Abstract. In the class $\mathbb{PR}$ of preradicals of the category of left R-modules R-Mod a new operation is defined and studied, namely the left quotient with respect to join. Some properties of this operation are shown, its compatibility with the lattice operations of $\mathbb{PR}$ (meet and join of preradicals), as well as the relations with some constructions in the "big" lattice $\mathbb{PR}$. Also some particular cases are examined.

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1 Introduction and preliminary facts

The work is concerned with the theory of radicals of modules ([1], [2], [3]) and is devoted to investigation of a new operation in the class of preradicals of a module category.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. A *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f : M \to M'$.

We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod, where the partial order relation is defined as follows:

$$r_1 \leq r_2 \overset{def}{\Leftrightarrow} r_1(M) \subseteq r_2(M) \text{ for every } M \in R\text{-Mod}.$$

In the class $\mathbb{PR}$ the following operations are defined ([1]):

1) the *meet* $\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha$ of the family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}} \subseteq \mathbb{PR}$:

$$\left(\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha\right)(M) \overset{def}{=} \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

2) the *join* $\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha$ of the family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}} \subseteq \mathbb{PR}$:

$$\left(\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha\right)(M) \overset{def}{=} \sum_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

3) the *product* $r \cdot s$ of preradicals $r, s \in \mathbb{PR}$:

$$(r \cdot s)(M) \overset{def}{=} r(s(M)),\ M \in R\text{-Mod};$$

4) the *coproduct* $r \# s$ of preradicals $r, s \in \mathbb{PR}$:

$$[(r \# s)(M)]/s(M) \overset{def}{=} r(M/s(M)), \; M \in R\text{-Mod}.$$

The class $\mathbb{PR}$ is a "big" complete lattice with respect to the operations meet and join.

We remark that in the book [1] the coproduct is denoted by $(r : s)$ and is defined by the rule $[(r : s)(M)]/r(M) = s(M/r(M))$, so $(r \# s) = (s : r)$.

The following properties of distributivity hold ([1]):

(1) $(\wedge r_\alpha) \cdot s = \wedge (r_\alpha \cdot s)$; (2) $(\vee r_\alpha) \cdot s = \vee (r_\alpha \cdot s)$;

(3) $(\wedge r_\alpha) \# s = \wedge (r_\alpha \# s)$; (4) $(\vee r_\alpha) \# s = \vee (r_\alpha \# s)$,

for every family of preradicals $\{r_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathbb{PR}$ and $s \in \mathbb{PR}$.

These relations permit to define some new operations in the class $\mathbb{PR}$. In the present work it is introduced and studied one of these operations, namely the left quotient with respect to join. The similar questions are discussed in [2], [6], [7] and [8].

Now we remind the principal types of preradicals. A preradical $r \in \mathbb{PR}$ is called:

- *an idempotent preradical*, if $r(r(M)) = r(M)$ for every $M \in R$-Mod (or if $r \cdot r = r$);
- *a radical*, if $r(M/r(M)) = 0$ for every $M \in R$-Mod (or if $r \# r = r$);
- *an idempotent radical*, if both previous conditions are fulfilled;
- *a pretorsion*, if $r(N) = N \bigcap r(M)$ for every $N \subseteq M \in R$-Mod;
- *a torsion*, if r is a radical and a pretorsion;
- *prime*, if $r \neq 1$ and for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \cdot t_2 \leq r$ implies either $t_1 \leq r$ or $t_2 \leq r$;
- *$\wedge$-prime*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 \leq r$ implies either $t_1 \leq r$ or $t_2 \leq r$;
- *irreducible*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \wedge t_2 = r$ implies $t_1 = r$ or $t_2 = r$.

The operations meet and join are commutative and associative, but the product and coproduct are only associative. By means of these operations four preradicals are obtained which are arranged in the following order:

$$r \cdot s \leq r \wedge s \leq r \vee s \leq r \# s$$

for every $r, s \in \mathbb{PR}$.

In the course of this work we will need the following facts and notions from general theory of preradicals (see [1]$-$[5]).

Lemma 1.1. *(Monotony of the product) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \cdot s_1 \leq r \cdot s_2$ and $s_1 \cdot r \leq s_2 \cdot r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.2. *(Monotony of the coproduct) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \# s_1 \leq r \# s_2$ and $s_1 \# r \leq s_2 \# r$ for every $r \in \mathbb{PR}$.* □

Lemma 1.3. *If r is a pretorsion, then $r \cdot s = r \wedge s$ for every $s \in \mathbb{PR}$.* □

Lemma 1.4. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

1) $(r \cdot s) \# t \geq (r \# t) \cdot (s \# t)$;

2) $(r \# s) \cdot t \leq (r \cdot t) \# (s \cdot t)$. □

Definition 1.1. The *annihilator* of preradical r is the preradical

$$a(r) = \vee \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot r = 0 \} .$$

Definition 1.2. The *pseudocomplement* of r in $\mathbb{PR}$ is a preradical $r^\perp \in \mathbb{PR}$ with the properties:

1) $r \wedge r^\perp = 0$;

2) If $s \in \mathbb{PR}$ is such that $s > r^\perp$, then $r \wedge s \neq 0$.

Lemma 1.5. *Each* $r \in \mathbb{PR}$ *has a unique pseudocomplement* $r^\perp$ *such that if* $s \in \mathbb{PR}$ *and* $r \wedge s = 0$, *then* $s \leq r^\perp$. □

Definition 1.3. The *supplement* of r in $\mathbb{PR}$ is a preradical $r^* \in \mathbb{PR}$ with the properties:

1) $r \vee r^* = 1$;

2) If $s \in \mathbb{PR}$ is such that $s < r^*$, then $r \vee s \neq 1$.

Lemma 1.6. *Let* $r \in \mathbb{PR}$ *and* r *possesses the supplement* r^*. *If* $s \in \mathbb{PR}$ *and* $r \vee s = 1$, *then* $s \geq r^*$. □

2 Left quotient with respect to join

We investigate the class of preradicals $\mathbb{PR}(\wedge, \vee, \cdot, \#)$ of category R-Mod provided with four operations defined above. Using these operations and the aforementioned properties of distributivity, some new inverse operations can be defined. One of them is defined and studied further.

Definition 2.1. Let $r, s \in \mathbb{PR}$. The *left quotient with respect to join of* r *by* s is defined as the greatest preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \cdot s \leq r$. We denote this preradical by $r \overset{\vee}{/}. s$.

We say that r is the *numerator* and s is the *denominator* of the quotient $r \overset{\vee}{/}. s$.

Now we mention the existence of the left quotient for every pair of preradicals.

Lemma 2.1. *For every* $r, s \in \mathbb{PR}$ *there exists the left quotient* $r \overset{\vee}{/}. s$ *with respect to join, and it can be presented in the form* $r \overset{\vee}{/}. s = \vee \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r \}$.

Proof. The family of preradicals $\{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r\}$ is not empty, because $0 \cdot s \leq$ $\leq r$. By the distributivity of the product with respect to the join of preradicals we obtain $\left(\bigvee_{r_\alpha \cdot s \leq r} r_\alpha\right) \cdot s = \bigvee_{r_\alpha \cdot s \leq r} (r_\alpha \cdot s)$. Since $r_\alpha \cdot s \leq r$ for preradicals r_α, we have $\bigvee_{r_\alpha \cdot s \leq r} (r_\alpha \cdot s) \leq r$, therefore $\left(\bigvee_{r_\alpha \cdot s \leq r} r_\alpha\right) \cdot s \leq r$. So the preradical $\bigvee_{r_\alpha \cdot s \leq r} r_\alpha$ is one of r_α, moreover it is the greatest among r_α with the property $r_\alpha \cdot s \leq r$. Therefore we have $\vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r\} = r \vee\!\!/. s$. □

From the proof of Lemma 2.1 it follows that $(r \vee\!\!/. s) \cdot s \leq r$ and we will use this relation often in continuation.

Lemma 2.2. *For every* $r, s \in \mathbb{PR}$ *we have* $r \vee\!\!/. s \geq r$.

Proof. Since $r \cdot s \leq r$ and $r \vee\!\!/. s = \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r\}$, it is clear that r is one of preradicals r_α. Therefore $r \leq \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r\}$, so $r \leq r \vee\!\!/. s$. □

The next two statements show the connection between the left quotient $r \vee\!\!/. s$ and the partial order $(\leq)$ in $\mathbb{PR}$.

Proposition 2.3. *(Monotony in the numerator) If* $r_1, r_2 \in \mathbb{PR}$ *and* $r_1 \leq r_2$*, then* $r_1 \vee\!\!/. s \,\leq\, r_2 \vee\!\!/. s$ *for every* $s \in \mathbb{PR}$ *.*

Proof. From Lemma 2.1 we have: $r_1 \vee\!\!/. s = \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r_1\}$ and $r_2 \vee\!\!/. s =$ $\vee \{r'_\alpha \in \mathbb{PR} \,|\, r'_\alpha \cdot s \leq r_2\}$. The relations $r_1 \leq r_2$ and $r_\alpha \cdot s \leq r_1$ imply $r_\alpha \cdot s \leq r_2$, so each r_α is one of the preradicals r'_α. This proves that $\vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r_1\} \leq$ $\vee \{r'_\alpha \in \mathbb{PR} \,|\, r'_\alpha \cdot s \leq r_2\}$, so $r_1 \vee\!\!/. s \,\leq\, r_2 \vee\!\!/. s$. □

Proposition 2.4. *(Antimonotony in the denominator) If* $s_1, s_2 \in \mathbb{PR}$ *and* $s_1 \leq s_2$*, then* $r \vee\!\!/. s_1 \geq r \vee\!\!/. s_2$ *for every* $s \in \mathbb{PR}$ *.*

Proof. From Lemma 2.1 we have:

$$r \vee\!\!/. s_1 = \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s_1 \leq r\}, \;\; r \vee\!\!/. s_2 = \vee \{r'_\alpha \in \mathbb{PR} \,|\, r'_\alpha \cdot s_2 \leq r\}.$$

If $s_1 \leq s_2$, then $r'_\alpha \cdot s_1 \leq r'_\alpha \cdot s_2$, but $r'_\alpha \cdot s_2 \leq r$, therefore $r'_\alpha \cdot s_1 \leq r$. So each preradical r'_α is one of the preradicals r_α and we obtain

$$\vee \{r'_\alpha \in \mathbb{PR} \,|\, r'_\alpha \cdot s_2 \leq r\} \leq \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s_1 \leq r\} \;,$$

i.e. $r \vee\!\!/. s_1 \geq r \vee\!\!/. s_2$. □

The following result is particulary useful in the further studies.

Proposition 2.5. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

$$r \geq t \cdot s \Leftrightarrow r \vee\!\!/. s \geq t \qquad .$$

Proof. $(\Rightarrow)$ Let $t \cdot s \leq r$. By Lemma 2.1 we have $r \vee\!\!/. s = \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r\}$. Then t is one of the preradicals r_α, therefore $t \leq \vee \{r_\alpha \in \mathbb{PR} \,|\, r_\alpha \cdot s \leq r\} = r \vee\!\!/. s$.

$(\Leftarrow)$ Let $t \leq r \vee\!\!/. s$. Then $t \cdot s \leq (r \vee\!\!/. s) \cdot s$ and by definition $(r \vee\!\!/. s) \cdot s \leq r$, therefore $t \cdot s \leq r$. □

Proposition 2.6. $(r \cdot s) \mathbin{\vee\!\!/.} s \geq r$ *for every preradicals* $r, s \in \mathbb{PR}$.

Proof. From Lemma 2.1 we have $(r \cdot s) \mathbin{\vee\!\!/.} s = \vee \{ t_\alpha \in \mathbb{PR} \mid t_\alpha \cdot s \leq r \cdot s \}$. Since $r \cdot s \leq r \cdot s$ we have that r is one of the preradicals t_α, therefore $r \leq \vee \{ t_\alpha \in \mathbb{PR} \mid t_\alpha \cdot s \leq r \cdot s \}$, i.e. $r \leq (r \cdot s) \mathbin{\vee\!\!/.} s$. □

Proposition 2.7. *For every* $r, s, t \in \mathbb{PR}$ *the following relations are true:*

1) $(r \mathbin{\vee\!\!/.} s) \mathbin{\vee\!\!/.} t = r \mathbin{\vee\!\!/.} (t \cdot s)$;

2) $(r \cdot s) \mathbin{\vee\!\!/.} t \geq r \cdot (s \mathbin{\vee\!\!/.} t)$.

Proof. 1) From Lemma 2.1 we have $r \mathbin{\vee\!\!/.} s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r \}$, $(r \mathbin{\vee\!\!/.} s) \mathbin{\vee\!\!/.} t = \vee \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \leq r \mathbin{\vee\!\!/.} s \}$ and $r \mathbin{\vee\!\!/.} (t \cdot s) = \vee \{ r'_\gamma \in \mathbb{PR} \mid r'_\gamma \cdot (t \cdot s) \leq r \}$.

($\leq$) If $t_\beta \cdot t \leq r \mathbin{\vee\!\!/.} s$, then from the monotony of the product $(t_\beta \cdot t) \cdot s \leq (r \mathbin{\vee\!\!/.} s) \cdot s$. By definition of the left quotient $(r \mathbin{\vee\!\!/.} s) \cdot s \leq r$, so $t_\beta \cdot (t \cdot s) = (t_\beta \cdot t) \cdot s \leq r$. This shows that each t_β is one of the preradicals r'_γ. Therefore $\vee \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot t \leq r \mathbin{\vee\!\!/.} s \} \leq \leq \vee \{ r'_\gamma \in \mathbb{PR} \mid r'_\gamma \cdot (t \cdot s) \leq r \}$, i.e $(r \mathbin{\vee\!\!/.} s) \mathbin{\vee\!\!/.} t \leq r \mathbin{\vee\!\!/.} (t \cdot s)$.

($\geq$) Let $r'_\gamma \cdot (t \cdot s) \leq r$. Then from the associativity of the product $(r'_\gamma \cdot t) \cdot s \leq r$, therefore any preradical of the form $(r'_\gamma \cdot t)$ is one of the preradicals r_α. This implies the following relation $(r'_\gamma \cdot t) \leq \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq r \} = r \mathbin{\vee\!\!/.} s$, which shows that each preradical r'_γ is one of the preradicals t_β. Therefore $\vee \{ r'_\gamma \in \mathbb{PR} \mid r'_\gamma \cdot (t \cdot s) \leq r \} \leq \leq \vee \{ t_\beta \in \mathbb{PR} \mid t_\beta \cdot s \leq r \mathbin{\vee\!\!/.} s \}$, i.e. $r \mathbin{\vee\!\!/.} (t \cdot s) \leq (r \mathbin{\vee\!\!/.} s) \mathbin{\vee\!\!/.} t$.

2) By the definition of left the quotient $s \geq (s \mathbin{\vee\!\!/.} t) \cdot t$. Then $r \cdot s \geq r \cdot [(s \mathbin{\vee\!\!/.} t) \cdot t] = [r \cdot (s \mathbin{\vee\!\!/.} t)] \cdot t$, and from Proposition 2.5 we obtain $(r \cdot s) \mathbin{\vee\!\!/.} t \geq r \cdot (s \mathbin{\vee\!\!/.} t)$. □

Proposition 2.8. *For every* $r, s, t \in \mathbb{PR}$ *the following relations hold:*

1) $(r \mathbin{\vee\!\!/.} t) \mathbin{\vee\!\!/.} (s \mathbin{\vee\!\!/.} t) \geq r \mathbin{\vee\!\!/.} s$;

2) $(r \cdot t) \mathbin{\vee\!\!/.} (s \cdot t) \geq r \mathbin{\vee\!\!/.} s$.

Proof. 1) From Proposition 2.5 the relation of this statement is equivalent to the relation $r \mathbin{\vee\!\!/.} t \geq (r \mathbin{\vee\!\!/.} s) \cdot (s \mathbin{\vee\!\!/.} t)$.

By definition of the left quotient we have $r \geq (r \mathbin{\vee\!\!/.} s) \cdot s$ and $s \geq (s \mathbin{\vee\!\!/.} t) \cdot t$, therefore $r \geq (r \mathbin{\vee\!\!/.} s) \cdot s \geq (r \mathbin{\vee\!\!/.} s) \cdot [(s \mathbin{\vee\!\!/.} t) \cdot t] = [(r \mathbin{\vee\!\!/.} s) \cdot (s \mathbin{\vee\!\!/.} t)] \cdot t$. Applying Proposition 2.5 we obtain $r \mathbin{\vee\!\!/.} t \geq (r \mathbin{\vee\!\!/.} s) \cdot (s \mathbin{\vee\!\!/.} t)$.

2) From Proposition 2.5 follows that the relation of this statement is equivalent to $r \cdot t \geq (r \mathbin{\vee\!\!/.} s) \cdot (s \cdot t)$. By definition of the left quotient we have $r \geq (r \mathbin{\vee\!\!/.} s) \cdot s$, therefore $r \cdot t \geq [(r \mathbin{\vee\!\!/.} s) \cdot s] \cdot t = (r \mathbin{\vee\!\!/.} s) \cdot (s \cdot t)$. □

Now we will indicate some relations between the left quotient with respect to join and the lattice operations of $\mathbb{PR}$.

Proposition 2.9. *(The distributivity of the left quotient* $r \mathbin{\vee\!\!/\!.} s$ *with respect to meet) Let* $s \in \mathbb{PR}$*. Then for every family of preradicals* $\{ r_\alpha | \, \alpha \in \mathfrak{A}\}$ *the following relation holds:*

$$\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right) \mathbin{\vee\!\!/\!.} s = \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s).$$

Proof. ($\geq$) By definition $r_\alpha \geq (r_\alpha \mathbin{\vee\!\!/\!.} s) \cdot s$, for every $\alpha \in \mathfrak{A}$. Then $\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \geq$ $\geq \bigwedge_{\alpha\in\mathfrak{A}} [(r_\alpha \mathbin{\vee\!\!/\!.} s) \cdot s]$. From the distributivity of the product of preradicals relative to meet it follows that $\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \geq \left[\bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s) \right] \cdot s$. Using Proposition 2.5 we obtain $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right) \mathbin{\vee\!\!/\!.} s \geq \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s)$.

($\leq$) From Lemma 2.1 we have $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right) \mathbin{\vee\!\!/\!.} s = \vee \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right\}$ and $\bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s) = \bigwedge_{\alpha\in\mathfrak{A}} \left(\bigvee_{r'_\gamma \cdot s \leq r_\alpha} r'_\gamma \right)$. Since $t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \leq r_\alpha$ for every $\alpha \in \mathfrak{A}$, we have $t_\beta \cdot s \leq r_\alpha$, so each preradical t_β is one of the preradicals r'_γ. This implies the relation $\vee \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right\} \leq \vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \cdot s \leq r_\alpha \}$ for every $\alpha \in \mathfrak{A}$, therefore $\vee \left\{ t_\beta \in \mathbb{PR} \,|\, t_\beta \cdot s \leq \bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right\} \leq \bigwedge_{\alpha\in\mathfrak{A}} (\vee \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \cdot s \leq r_\alpha \})$, which means that $\left(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha \right) \mathbin{\vee\!\!/\!.} s \leq \bigwedge_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s)$. □

Proposition 2.10. *In the class* $\mathbb{PR}$ *the following relations are true:*

1) $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \right) \mathbin{\vee\!\!/\!.} s \geq \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s)$;

2) $r \mathbin{\vee\!\!/\!.} \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha \right) \geq \bigvee_{\alpha\in\mathfrak{A}} (r \mathbin{\vee\!\!/\!.} s_\alpha)$;

3) $r \mathbin{\vee\!\!/\!.} \left(\bigvee_{\alpha\in\mathfrak{A}} s_\alpha \right) \leq \bigwedge_{\alpha\in\mathfrak{A}} (r \mathbin{\vee\!\!/\!.} s_\alpha)$.

Proof. 1) By the definition of the left quotient we have $r_\alpha \geq (r_\alpha \mathbin{\vee\!\!/\!.} s) \cdot s$ for every $\alpha \in \mathfrak{A}$, therefore $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \geq \bigvee_{\alpha\in\mathfrak{A}} [(r_\alpha \mathbin{\vee\!\!/\!.} s) \cdot s]$. From the distributivity of the product of preradicals relative to join it follows that $\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \geq \left[\bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s) \right] \cdot s$ and using Proposition 2.5 we obtain $\left(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha \right) \mathbin{\vee\!\!/\!.} s \geq \bigvee_{\alpha\in\mathfrak{A}} (r_\alpha \mathbin{\vee\!\!/\!.} s)$.

2) For every $\alpha \in \mathfrak{A}$ we have $\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha \leq s_\alpha$. From the antimonotony in the denominator it follows that $r \mathbin{\vee\!\!/\!.} \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha \right) \geq r \mathbin{\vee\!\!/\!.} s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \mathbin{\vee\!\!/\!.} \left(\bigwedge_{\alpha\in\mathfrak{A}} s_\alpha \right) \geq \bigvee_{\alpha\in\mathfrak{A}} (r \mathbin{\vee\!\!/\!.} s_\alpha)$.

3) For every $\alpha \in \mathfrak{A}$ we have $\bigvee\limits_{\alpha \in \mathfrak{A}} s_\alpha \geq s_\alpha$. From the antimonotony in the denominator it follows that $r \overset{\vee}{/}. \left(\bigvee\limits_{\alpha \in \mathfrak{A}} s_\alpha \right) \leq r \overset{\vee}{/}. s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \overset{\vee}{/}. \left(\bigvee\limits_{\alpha \in \mathfrak{A}} s_\alpha \right) \leq \bigwedge\limits_{\alpha \in \mathfrak{A}} (r \overset{\vee}{/}. s_\alpha)$. □

3 The left quotient $r \overset{\vee}{/}. s$ in particular cases

In this section we will show some particular cases of left quotient with respect to join, its relations with some constructions in the "big" lattice $\mathbb{PR}$ and its connection with certain types of preradicals (prime, $\wedge$-prime, irreducible), as well as the arrangement (relative position) of preradicals obtained by left quotient.

Proposition 3.1. *For every preradicals $r, s \in \mathbb{PR}$ the following conditions are equivalent:*

1) $r \geq s$;
2) $r \overset{\vee}{/}. s = 1$.

Proof. 1) ⇒ 2) Let $r \geq s$, then $r \geq 1 \cdot s$ and from Proposition 2.5 we obtain $r \overset{\vee}{/}. s \geq 1$, therefore $r \overset{\vee}{/}. s = 1$.

2) ⇒ 1) Let $r \overset{\vee}{/}. s = 1$. By the definition of the left quotient we have $(r \overset{\vee}{/}. s) \cdot s \leq r$, so $1 \cdot s \leq r$, i.e $s \leq r$. □

Proposition 3.2. *Let $r, s \in \mathbb{PR}$. Then:*

1) $0 \overset{\vee}{/}. s = a(s)$ *(see Definition 1.1);*
2) $r \overset{\vee}{/}. 1 = r$.

Proof. From the definition of left quotient we have:

1) $0 \overset{\vee}{/}. s = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s \leq 0 \} = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s = 0 \} = a(s)$;
2) $r \overset{\vee}{/}. 1 = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot 1 \leq r \} = \vee \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \leq r \} = r$. □

From Propositions 3.1 and 3.2 it follows the following particular cases:

(1) $0 \overset{\vee}{/}. 0 = a(0) = 1$;	(2) $r \overset{\vee}{/}. r = 1$, $\forall r \in \mathbb{PR}$;
(3) $1 \overset{\vee}{/}. s = 1$, $\forall s \in \mathbb{PR}$;	(4) $1 \overset{\vee}{/}. 1 = 1$.

As in Proposition 3.1 $(r \overset{\vee}{/}. r) \cdot r = 1 \cdot r = r$, for every $r \in \mathbb{PR}$.

Moreover, the distributivity of product of preradicals relative to the join implies $a(s) \cdot s = \left(\bigvee\limits_{r_\alpha \cdot s = 0} r_\alpha \right) \cdot s = \bigvee\limits_{r_\alpha \cdot s = 0} (r_\alpha \cdot s) = 0$ for every $s \in \mathbb{PR}$.

In continuation we will discuss the question of the relations between the annihilator $a(r)$ and some constructions in the "big" lattice $\mathbb{PR}$ such as pseudocomplement and supplement.

Proposition 3.3. *For every preradical $s \in \mathbb{PR}$ we have $a(s) \geq s^\perp$.*

Proof. By the definition of the annihilator $a(s) = \vee\{r_\alpha \mid r_\alpha \cdot s = 0\}$. The pseudo-complement $s^\perp$ of the preradical s, by the definition, has the property $s^\perp \wedge s = 0$. Since $s^\perp \cdot s \leq s^\perp \wedge s = 0$, we obtain $s^\perp \cdot s = 0$. So $s^\perp$ is one of the preradicals r_α, therefore $s^\perp \leq \vee\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s = 0\}$, i.e. $s^\perp \leq a(s)$. □

Moreover, from Proposition 2.3 we have $r \vee\!\!\!/.\, s \geq 0 \vee\!\!\!/.\, s = a(s)$, therefore $r \vee\!\!\!/.\, s \geq s^\perp$.

Proposition 3.4. *Let $s \in \mathbb{PR}$ and s has the supplement s^*. Then $a(s) \leq s^*$.*

Proof. By definition $a(s) = \vee\{r_\alpha \in \mathbb{PR} \mid r_\alpha \cdot s = 0\}$. The supplement s^* of s from the definition has the property $s^* \vee s = 1$. Since $s \# s^* \geq s \vee s^* = 1$, we obtain $s \# s^* = 1$. We have that $a(s) \cdot s = 0$, so $s^* = 0 \# s^* = (a(s) \cdot s) \# s^*$. From Lemma 1.4 $(a(s) \cdot s) \# s^* \geq (a(s) \# s^*) \cdot (s \# s^*) = (a(s) \# s^*) \cdot 1 = a(s) \# s^*$, therefore $s^* \geq a(s) \# s^*$. But $a(s) \# s^* \geq a(s)$ and so $s^* \geq a(s)$. □

Furtheremore, we have $s^* \geq a(s) \# s^*$ and $a(s) \# s^* \geq s^*$, so $s^* = a(s) \# s^*$.

In the next two statements it is shown when the cancellation properties hold (see Proposition 2.6).

Proposition 3.5. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \cdot s) \vee\!\!\!/.\, s$.
2) $r = t \vee\!\!\!/.\, s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) Let $r = (r \cdot s) \vee\!\!\!/.\, s$. Then $r = t \vee\!\!\!/.\, s$ with $t = r \cdot s$.

2) $\Rightarrow$ 1) Let $r = t \vee\!\!\!/.\, s$ for some preradical t. By the definition of the left quotient we have $(t \vee\!\!\!/.\, s) \cdot s \leq t$. From Proposition 2.3 we obtain $[(t \vee\!\!\!/.\, s) \cdot s] \vee\!\!\!/.\, s \leq t \vee\!\!\!/.\, s$. But from Proposition 2.6 $[(t \vee\!\!\!/.\, s) \cdot s] \vee\!\!\!/.\, s \geq t \vee\!\!\!/.\, s$, therefore $[(t \vee\!\!\!/.\, s) \cdot s] \vee\!\!\!/.\, s = t \vee\!\!\!/.\, s$. Since $t \vee\!\!\!/.\, s = r$, we have $(r \cdot s) \vee\!\!\!/.\, s = r$. □

Proposition 3.6. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \vee\!\!\!/.\, s) \cdot s$.
2) $r = t \cdot s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) Let $r = (r \vee\!\!\!/.\, s) \cdot s$. Then $r = t \cdot s$ with $t = r \vee\!\!\!/.\, s$.

2) $\Rightarrow$ 1) Let $r = t \cdot s$ for some preradical t. By Proposition 2.6 we have $(t \cdot s) \vee\!\!\!/.\, s \geq t$. From the monotony of the product it follows that $[(t \cdot s) \vee\!\!\!/.\, s] \cdot s \geq$ $\geq t \cdot s$. But from the definition of the left quotient $[(t \cdot s) \vee\!\!\!/.\, s] \cdot s \leq t \cdot s$, therefore $[(t \cdot s) \vee\!\!\!/.\, s] \cdot s = t \cdot s$. Since $t \cdot s = r$, we have $(r \vee\!\!\!/.\, s) \cdot s = r$. □

Now we will show the behaviour of the left quotient $r \vee\!\!\!/.\, s$ in the cases of some types of preradicals (prime, $\wedge$-prime, irreducible).

Proposition 3.7. *The preradical r is prime if and only if for every preradical s we have $r \vee\!\!\!/.\, s = 1$ or $r \vee\!\!\!/.\, s = r$.*

Proof. ($\Rightarrow$) Let $r \neq 1$. By definition $(r \mathbin{\vee\!\!/.} s) \cdot s \leq r$ and if r is prime, then we have $r \mathbin{\vee\!\!/.} s \leq r$ or $s \leq r$. If $r \mathbin{\vee\!\!/.} s \leq r$, then by Lemma 2.2 $r \mathbin{\vee\!\!/.} s \geq r$, therefore $r \mathbin{\vee\!\!/.} s = r$. If $s \leq r$, then from Proposition 3.1 we have $r \mathbin{\vee\!\!/.} s = 1$.

($\Leftarrow$) Let $t_1 \cdot t_2 \leq r$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. From Proposition 2.5 we obtain $t_1 \leq r \mathbin{\vee\!\!/.} t_2$. For the preradical t_2 from the condition of this proposition we have $r \mathbin{\vee\!\!/.} t_2 = 1$ or $r \mathbin{\vee\!\!/.} t_2 = r$. If $r \mathbin{\vee\!\!/.} t_2 = 1$, then from Proposition 3.1 it follows that $t_2 \leq r$. If $r \mathbin{\vee\!\!/.} t_2 = r$, then $t_1 \leq r \mathbin{\vee\!\!/.} t_2 = r$. So for every $t_1, t_2 \in \mathbb{PR}$ with $t_1 \cdot t_2 \leq r$ we have $t_1 \leq r$ or $t_2 \leq r$, which means that the preradical r is prime. □

Proposition 3.8. *If the preradical r is $\wedge$-prime, then the quotient $r \mathbin{\vee\!\!/.} s$ is $\wedge$-prime for every $s \in \mathbb{PR}$.*

Proof. Suppose that $t_1 \wedge t_2 \leq r \mathbin{\vee\!\!/.} s$. Then from Proposition 2.5 we obtain $(t_1 \wedge t_2) \cdot s \leq r$. From the distributivity of the product of preradicals relative to meet we have $(t_1 \cdot s) \wedge (t_2 \cdot s) \leq r$. If r is $\wedge$-prime, then $t_1 \cdot s \leq r$ or $t_2 \cdot s \leq r$. From Proposition 2.5 we obtain that $t_1 \leq r \mathbin{\vee\!\!/.} s$ or $t_2 \leq r \mathbin{\vee\!\!/.} s$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \wedge t_2 \leq r \mathbin{\vee\!\!/.} s$ we have $t_1 \leq r \mathbin{\vee\!\!/.} s$ or $t_2 \leq r \mathbin{\vee\!\!/.} s$, which means that the preradical $r \mathbin{\vee\!\!/.} s$ is $\wedge$-prime. □

Proposition 3.9. *Let $r, s \in \mathbb{PR}$ and $r = t \cdot s$ for some preradical $t \in \mathbb{PR}$. If the preradical r is irreducible, then the preradical $r \mathbin{\vee\!\!/.} s$ is irreducible.*

Proof. Let for some preradicals $t_1, t_2 \in \mathbb{PR}$ we have $t_1 \wedge t_2 = r \mathbin{\vee\!\!/.} s$. If $r = t \cdot s$ for some preradical t, then by Proposition 3.5 $r = (r \mathbin{\vee\!\!/.} s) \cdot s$, so $r = (t_1 \wedge t_2) \cdot s$. From the distributivity of the product of preradicals with respect to meet we obtain $r = (t_1 \cdot s) \wedge (t_2 \cdot s)$. If r is irreducible, then $t_1 \cdot s = r$ or $t_2 \cdot s = r$.

If $t_1 \cdot s = r$, then from Proposition 2.5 we have $t_1 \leq r \mathbin{\vee\!\!/.} s$. But $t_1 \geq r \mathbin{\vee\!\!/.} s$, because $t_1 \wedge t_2 = r \mathbin{\vee\!\!/.} s$, therefore $t_1 = r \mathbin{\vee\!\!/.} s$.

If $t_2 \cdot s = r$, then similarly we obtain $t_2 = r \mathbin{\vee\!\!/.} s$.

So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \wedge t_2 = r \mathbin{\vee\!\!/.} s$ we have $t_1 = r \mathbin{\vee\!\!/.} s$ or $t_2 = r \mathbin{\vee\!\!/.} s$, which means that the preradical $r \mathbin{\vee\!\!/.} s$ is irreducible. □

The operation of the left quotient $r \mathbin{\vee\!\!/.} s$ implies the following arrangement of associated preradicals.

Proposition 3.10. *For every $r, s, t \in \mathbb{PR}$ the following relations are true:*

1) $r \mathbin{\vee\!\!/.} s = (r \wedge s) \mathbin{\vee\!\!/.} s$;

2) $(r \mathbin{\vee\!\!/.} s) \cdot s \leq r \wedge s$.

Proof. 1) From Proposition 2.9 we have $(r \wedge s) \mathbin{\vee\!\!/.} s = (r \mathbin{\vee\!\!/.} s) \wedge (s \mathbin{\vee\!\!/.} s)$, but $s \mathbin{\vee\!\!/.} s = 1$, so $(r \wedge s) \mathbin{\vee\!\!/.} s = (r \mathbin{\vee\!\!/.} s) \wedge 1 = r \mathbin{\vee\!\!/.} s$.

Moreover, since $r \cdot s \leq r \wedge s$, from Proposition 2.3 we obtain

$$(r \cdot s) \mathbin{\vee\!\!/.} s \leq (r \wedge s) \mathbin{\vee\!\!/.} s = r \mathbin{\vee\!\!/.} s.$$

2) By 1) we have $r \mathbin{\vee\!\!/.} s = (r \wedge s) \mathbin{\vee\!\!/.} s$ and from the monotony of the product of preradicals we obtain $(r \mathbin{\vee\!\!/.} s) \cdot s = ((r \wedge s) \mathbin{\vee\!\!/.} s) \cdot s$. From the definition of the left quotient we have $((r \wedge s) \mathbin{\vee\!\!/.} s) \cdot s \leq r \wedge s$, therefore $(r \mathbin{\vee\!\!/.} s) \cdot s \leq r \wedge s$. □

Corollary 3.11. *1) For every preradicals* $r, s \in \mathbb{PR}$ *the following relations hold:*

$$r \cdot s \leq (r \overset{\vee}{/} s) \cdot s \leq r \wedge s \leq r \leq (r \cdot s) \overset{\vee}{/} s \leq r \overset{\vee}{/} s;$$

2) If r *is a pretorsion, then*

$$r \cdot s = (r \overset{\vee}{/} s) \cdot s = r \wedge s \leq r \leq (r \cdot s) \overset{\vee}{/} s = r \overset{\vee}{/} s$$

for every $s \in \mathbb{PR}$. □

In conclusion we can say that in the class $\mathbb{PR}$ of the category R-Mod there is defined a new operation - left quotient with respect to join, which possesses a series of properties connected with the four operations of the class $\mathbb{PR}$. This new operation is concordant with a series of notions from the theory of radicals.

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References

[1] Bican L., Kepka T., Nemec P. *Rings, modules and preradicals.* Marcel Dekker, New York, 1982.

[2] Golan J. S. *Linear topologies on a ring.* Logman Sci. Techn., New York, 1987.

[3] Kashu A. I. *Radicals and torsions in modules.* Kishinev, Ştiinţa, 1983 (in Russian).

[4] Raggi F., Rios J., Rincon H., Fernandes-Alonso R., Signoret C. *The lattice structure of preradicals.* Commun. Algebra, 2002, **1**, No. 3, 1533–1544.

[5] Raggi F., Rios J., Rincon H., Fernandes-Alonso R., Signoret C. *The lattice structure of preradicals II.* J. of Algebra and its Applications, 2002, **1**, No. 2, 201–214.

[6] Kashu A. I. *On some operations in the lattice of submodules determined by preradicals.* Bul. Acad. Ştiinţe Repub. Moldova, Matematica, 2011, No. 2(66), 5–16.

[7] Raggi F., Rios J., Rincon H., Fernandes-Alonso R., Signoret C. *Prime and irreducible preradicals.* J. of Algebra and its Applications, 2005, **4**, No. 4, 451–466.

[8] Kashu A. I. *On inverse operations in the lattices of submodules.* Algebra and Discrete Mathematics, 2012, **13**, No. 2, 273–288.

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