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On the inverse operations in the class of preradicals of a module category, II

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Abstract. In the present work a new operation, called left coquotient with respect to meet, in the class of preradicals $\mathbb{PR}$ of the category R-Mod of left R-modules is defined and investigated. It is dual to the studied earlier left quotient with respect to join [2]. Main properties of this operation and relations with lattice operations in $\mathbb{PR}$ are shown. Connections with some constructions in the large complete lattice $\mathbb{PR}$ are studied and some particular cases are mentioned.

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1 Introduction and preliminary facts

This work is devoted to the theory of radicals of modules ([1], [4]-[7]) and contains the investigation of a new operation in the class of preradicals of a module category.

Let R be a ring with unity and R-Mod be the category of unitary left R-modules. We remind that a *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. r associates to every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $f(r(M)) \subseteq r(M')$ for every R-morphism $f: M \to M'$.

We denote by $\mathbb{PR}$ the class of all preradicals of the category R-Mod. In this class four operation are defined [4]:

1) the *meet* $\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}}$:
$$\left(\underset{\alpha\in\mathfrak{A}}{\wedge} r_\alpha\right)(M) \overset{def}{=} \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

2) the *join* $\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha$ of a family of preradicals $\{r_\alpha\}_{\alpha\in\mathfrak{A}}$:
$$\left(\underset{\alpha\in\mathfrak{A}}{\vee} r_\alpha\right)(M) \overset{def}{=} \sum_{\alpha\in\mathfrak{A}} r_\alpha(M),\ M \in R\text{-Mod};$$

3) the *product* $r \cdot s$ of preradicals $r, s \in \mathbb{PR}$:
$$(r \cdot s)(M) \overset{def}{=} r(s(M)),\ M \in R\text{-Mod}\ ;$$

4) the *coproduct* $r \# s$ of preradicals $r, s \in \mathbb{PR}$:
$$[(r \# s)(M)]/s(M) \overset{def}{=} r(M/s(M)\ ),\ M \in R\text{-Mod}.$$

In the class $\mathbb{PR}$ the partial order relation " $\leq$ " is defined by the rule:

$$r_1 \leq r_2 \overset{def}{\Leftrightarrow} r_1(M) \subseteq r_2(M) \ \text{ for every } \ M \in R\text{-Mod}.$$

The class $\mathbb{PR}$ is a large complete lattice with respect to the operations of meet and join.

We remark that in the book [4] the coproduct is denoted by $(r : s)$ and is defined by the rule $[(r : s)(M)]/r(M) = s(M/r(M))$, so $(r \# s) = (s : r)$.

The following properties of distributivity hold [4]:

(1) $(\wedge r_\alpha) \cdot s = \wedge (r_\alpha \cdot s)$; (2) $(\vee r_\alpha) \cdot s = \vee (r_\alpha \cdot s)$;

(3) $(\wedge r_\alpha) \# s = \wedge (r_\alpha \# s)$; (4) $(\vee r_\alpha) \# s = \vee (r_\alpha \# s)$

for every family $\{r_\alpha\}_{\alpha \in \mathfrak{A}} \subseteq \mathbb{PR}$ and $s \in \mathbb{PR}$.

Using these relations some new inverse operations can be defined in the class $\mathbb{PR}$. One of them, the left quotient of product with respect to join, was defined and investigated in [2]. In this work we will study another inverse operation, namely the left coquotient of coproduct with respect to meet. In the case of pretorsions it was investigated by J. S. Golan by other methods in [1] (see [3]). Similar questions are discussed in [8], [9] and [10].

Now we remind the principal types of preradicals. A preradical $r \in \mathbb{PR}$ is called:

- *idempotent preradical*, if $r(r(M)) = r(M)$ for every $M \in R$-Mod (or if $r \cdot r = r$);
- *radical*, if $r(M/r(M)) = 0$ for every $M \in R$-Mod (or if $r \# r = r$);
- *idempotent radical*, if both previous conditions are fulfilled;
- *pretorsion (hereditary preradical)*, if $r(N) = N \bigcap r(M)$ for every $N \subseteq M$, $M \in R$-Mod;
- *cohereditary*, if $r(M/N) = (r(M) + N)/N$, for every $N \subseteq M \in R$-Mod;
- *torsion*, if r is a hereditary radical;
- *coprime*, if $r \neq 0$ and for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \# t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [9];
- $\vee$-*coprime*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 \geq r$ implies $t_1 \geq r$ or $t_2 \geq r$ [9];
- *coirreducible*, if for any $t_1, t_2 \in \mathbb{PR}$, $t_1 \vee t_2 = r$ implies $t_1 = r$ or $t_2 = r$ [9].

The operations of meet and join are commutative and associative, while the operations of product and coproduct are associative. By means of these operations four preradicals are obtained which are arranged in the following order:

$$r \cdot s \leq r \wedge s \leq r \vee s \leq r \# s$$

for every $r, s \in \mathbb{PR}$.

During this work we will use the following facts and notions from general theory of preradicals (see [4]–[7]).

Lemma 1.1. *(Monotony of the product) For any $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ implies that $r \cdot s_1 \leq r \cdot s_2$ and $s_1 \cdot r \leq s_2 \cdot r$ for every $r \in \mathbb{PR}$.* $\square$

Lemma 1.2. *(Monotony of the coproduct) For any* $s_1, s_2 \in \mathbb{PR}$, $s_1 \leq s_2$ *implies that* $r \# s_1 \leq r \# s_2$ *and* $s_1 \# r \leq s_2 \# r$ *for every* $r \in \mathbb{PR}$. $\square$

Lemma 1.3. *If the preradical* r *is cohereditary, then* $r \# s = r \vee s$ *for every* $s \in \mathbb{PR}$. $\square$

Lemma 1.4. *For every* $r, s, t \in \mathbb{PR}$ *we have:*

1) $(r \cdot s) \# t \geq (r \# t) \cdot (s \# t)$;

2) $(r \# s) \cdot t \leq (r \cdot t) \# (s \cdot t)$. $\square$

Definition 1.1. The *totalizer* of preradical r is the preradical

$$t\,(r) = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \# r = 1 \} \ .$$

Definition 1.2. The *pseudocomplement* of r in $\mathbb{PR}$ is a preradical $r^\perp \in \mathbb{PR}$ with the properties:

1) $r \wedge r^\perp = 0$;

2) If $s \in \mathbb{PR}$ is such that $s > r^\perp$, then $r \wedge s \neq 0$.

Lemma 1.5. *Each* $r \in \mathbb{PR}$ *has a unique pseudocomplement* $r^\perp$ *such that if* $s \in \mathbb{PR}$ *and* $r \wedge s = 0$, *then* $s \leq r^\perp$. $\square$

Definition 1.3. The *supplement* of r in $\mathbb{PR}$ is a preradical $r^* \in \mathbb{PR}$ with the properties:

1) $r \vee r^* = 1$;

2) If $s \in \mathbb{PR}$ is such that $s < r^*$, then $r \vee s \neq 1$.

Lemma 1.6. *Let* $r \in \mathbb{PR}$ *and* r *possesses the supplement* r^*. *If* $s \in \mathbb{PR}$ *and* $r \vee s = 1$, *then* $s \geq r^*$. $\square$

2 Left coquotient with respect to meet

Now we introduce and investigate the inverse operation of coproduct with respect to meet in the class of preradicals $\mathbb{PR}$ of category R-Mod.

Definition 2.1. Let $r, s \in \mathbb{PR}$. The *left coquotient with respect to meet of* r *by* s is defined as the least preradical among $r_\alpha \in \mathbb{PR}$ with the property $r_\alpha \# s \geq r$. We denote this preradical by $r \,{}^{\wedge}\!\!/_{\#}\, s$.

We will call r the *numerator* and s the *denominator* of the coquotient $r \,{}^{\wedge}\!\!/_{\#}\, s$.

Now we mention the existence of the left coquotient for every pair of preradicals.

Lemma 2.1. *For every* $r, s \in \mathbb{PR}$ *there exists the left coquotient* $r \,{}^{\wedge}\!\!/_{\#}\, s$ *with respect to meet, and it can be presented in the form* $r \,{}^{\wedge}\!\!/_{\#}\, s = \wedge \{ r_\alpha \in \mathbb{PR} \,|\, r_\alpha \# s \geq r \}$.

Proof. Since $1 \# s \geq r$ for every $s \in \mathbb{PR}$, the family of preradicals $\{r_\alpha \mid r_\alpha \# s \geq r\}$ is not empty. By the distributivity of coproduct with respect to meet of preradicals we have $\left(\underset{r_\alpha \# s \geq r}{\wedge} r_\alpha\right) \# s = \underset{r_\alpha \# s \geq r}{\wedge} (r_\alpha \# s)$. Since $r_\alpha \# s \geq r$ for every preradical r_α it follows that $\underset{r_\alpha \# s \geq r}{\wedge} (r_\alpha \# s) \geq r$, i.e. $\left(\underset{r_\alpha \# s \geq r}{\wedge} r_\alpha\right) \# s \geq r$. Therefore the preradical $\underset{r_\alpha \# s \geq r}{\wedge} r_\alpha$ is one of r_α and it is the least among r_α with the property $r_\alpha \# s \geq r$. So $r \hat{/}_{\#} s = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$. $\square$

Moreover, from the proof of Lemma 2.1 it follows that $(r \hat{/}_{\#} s) \# s \geq r$. We will often use this relation futher.

Lemma 2.2. *For every $r, s \in \mathbb{PR}$ we have $r \hat{/}_{\#} s \leq r$.*

Proof. By Lemma 2.1 $r \hat{/}_{\#} s = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$. Since $r \# s \geq r$ it follows that r is one of preradicals r_α. Therefore $r \geq \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$, i.e. $r \geq r \hat{/}_{\#} s$. $\square$

Now we indicate the behaviour of the left coquotient with respect to the order relation $(\leq)$ of $\mathbb{PR}$.

Proposition 2.3. *(Monotony in the numerator) If $r_1, r_2 \in \mathbb{PR}$ and $r_1 \leq r_2$, then $r_1 \hat{/}_{\#} s \leq r_2 \hat{/}_{\#} s$ for every $s \in \mathbb{PR}$.*

Proof. From Lemma 2.1 we have $r_1 \hat{/}_{\#} s = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r_1\}$ and $r_2 \hat{/}_{\#} s = \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s \geq r_2 \right\}$. The relations $r_1 \leq r_2$ and $r'_\beta \# s \geq r_2$ imply $r'_\beta \# s \geq r_1$, so each r'_β is one of preradicals r_α. This proves that $\wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r_1\} \leq \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s \geq r_2 \right\}$, so $r_1 \hat{/}_{\#} s \leq r_2 \hat{/}_{\#} s$. $\square$

Proposition 2.4. *(Antimonotony in the denominator) If $s_1, s_2 \in \mathbb{PR}$ and $s_1 \leq s_2$, then $r \hat{/}_{\#} s_1 \geq r \hat{/}_{\#} s_2$ for every $s \in \mathbb{PR}$.*

Proof. From Lemma 2.1 we have $r \hat{/}_{\#} s_1 = \wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s_1 \geq r\}$ and $r \hat{/}_{\#} s_2 = \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s_2 \geq r \right\}$. Let $s_1 \leq s_2$. Then from the monotony of coproduct we have $r_\alpha \# s_1 \leq r_\alpha \# s_2$. Since $r_\alpha \# s_1 \geq r$, we obtain $r_\alpha \# s_2 \geq r$. So each preradical r_α is one of preradicals r'_β, therefore

$$\wedge \{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s_1 \geq r\} \geq \wedge \left\{ r'_\beta \in \mathbb{PR} \,\middle|\, r'_\beta \# s_2 \geq r \right\},$$

i.e. $r \hat{/}_{\#} s_1 \geq r \hat{/}_{\#} s_2$. $\square$

The following fact is very useful for the further investigations.

Proposition 2.5. *For every $r, s, t \in \mathbb{PR}$ we have:*

$$r \leq t \# s \Leftrightarrow r \hat{/}_{\#} s \leq t.$$

Proof. ($\Rightarrow$) By Lemma 2.1 $r \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\}$. If $t \# s \geq r$, then t is one of preradicals r_α, therefore $t \geq \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq r\} = r \,{}^{\wedge}\!\!/_{\#}\, s$.

($\Leftarrow$) Let $t \geq r \,{}^{\wedge}\!\!/_{\#}\, s$. From the monotony of coproduct $t \# s \geq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$ and by definition of left coquotient we have $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$, therefore $t \# s \geq r$. $\square$

In continuation we show some properties of the studied operation.

Proposition 2.6. *For every preradicals $r, s \in \mathbb{PR}$ we have:*

$$(r \# s) \,{}^{\wedge}\!\!/_{\#}\, s \leq r.$$

Proof. From Lemma 2.1 we have $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, s = \wedge\{t_\alpha \in \mathbb{PR} \mid t_\alpha \# s \geq r \# s\}$. Since $r \# s \geq r \# s$, the preradical r is one of preradicals t_α, therefore we obtain $r \geq \wedge\{t_\alpha \in \mathbb{PR} \mid t_\alpha \# s \leq r \# s\}$, i.e. $r \geq (r \# s) \,{}^{\wedge}\!\!/_{\#}\, s$. $\square$

Proposition 2.7. *For every $r, s, t \in \mathbb{PR}$ the following relations are true:*

1) $(r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t \;=\; r \,{}^{\wedge}\!\!/_{\#}\, (t \# s)$;

2) $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, t \;\leq\; r \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$.

Proof. 1) From Lemma 2.1 we have $r \,{}^{\wedge}\!\!/_{\#}\, (t \# s) \;=\; \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# (t \# s) \geq r\}$ and $(r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t = \wedge\{t_\beta \in \mathbb{PR} \mid t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s\}$.

($\leq$) Let $r_\alpha \# (t \# s) \geq r$. Then $(r_\alpha \# t) \# s \geq r$ and from Proposition 2.5 we obtain $r_\alpha \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s$. So any preradical r_α is one of preradicals t_β, therefore we obtain $\wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# (t \# s) \geq r\} \geq \wedge\{t_\beta \in \mathbb{PR} \mid t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s\}$, i.e. $r \,{}^{\wedge}\!\!/_{\#}\, (t \# s) \geq (r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t$.

($\geq$) Let $t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s$. Using the monotony of coproduct we obtain $(t_\beta \# t) \# s \geq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$, but from the definition of left coquotient $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$, so $t_\beta \# (t \# s) = (t_\beta \# t) \# s \geq r$. This shows that each preradical t_β is one of preradicals r_α, therefore $\wedge\{t_\beta \in \mathbb{PR} \mid t_\beta \# t \geq r \,{}^{\wedge}\!\!/_{\#}\, s\} \geq \wedge\{r_\alpha \in \mathbb{PR} \mid r_\alpha \# (t \# s) \geq r\}$, i.e $(r \,{}^{\wedge}\!\!/_{\#}\, s) \,{}^{\wedge}\!\!/_{\#}\, t \;\geq\; r \,{}^{\wedge}\!\!/_{\#}\, (t \# s)$.

2) By definition of left coquotient $s \;\leq\; (s \,{}^{\wedge}\!\!/_{\#}\, t) \# t$. Using the monotony of coproduct we have $r \# s \leq r \# [(s \,{}^{\wedge}\!\!/_{\#}\, t) \# t] = [r \# (s \,{}^{\wedge}\!\!/_{\#}\, t)] \# t$, and from Proposition 2.5 we obtain $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, t \;\leq\; r \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$. $\square$

Proposition 2.8. *For every $r, s, t \in \mathbb{PR}$ the following relations hold:*

1) $(r \,{}^{\wedge}\!\!/_{\#}\, t) \,{}^{\wedge}\!\!/_{\#}\, (s \,{}^{\wedge}\!\!/_{\#}\, t) \leq r \,{}^{\wedge}\!\!/_{\#}\, s$;

2) $(r \# t) \,{}^{\wedge}\!\!/_{\#}\, (s \# t) \leq r \,{}^{\wedge}\!\!/_{\#}\, s$.

Proof. 1) From Proposition 2.5 the relation of this statement is equivalent to the relation $r \,{}^{\wedge}\!\!/_{\#}\, t \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$.

By definition of left coquotient $r \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$ and $s \leq (s \,{}^{\wedge}\!\!/_{\#}\, t) \# t$, therefore from the monotony and the associativity of coproduct we obtain $r \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# [(s \,{}^{\wedge}\!\!/_{\#}\, t) \# t] \;=\; [(r \,{}^{\wedge}\!\!/_{\#}\, s) \# (s \,{}^{\wedge}\!\!/_{\#}\, t)] \# t$. Applying Proposition 2.5 we have $r \,{}^{\wedge}\!\!/_{\#}\, t \leq (r \,{}^{\wedge}\!\!/_{\#}\, s) \# (s \,{}^{\wedge}\!\!/_{\#}\, t)$.

2) From Proposition 2.5 the relation of this statement is equivalent to the relation $r \# t \leq (r \,{}^{\wedge}\!/_{\#}\, s) \# (s \# t)$.

By definition of left coquotient $r \leq (r \,{}^{\wedge}\!/_{\#}\, s) \# s$. Using the monotony of coproduct we obtain $r \# t \leq [(r \,{}^{\wedge}\!/_{\#}\, s) \# s\,] \# t = (r \,{}^{\wedge}\!/_{\#}\, s) \# (s \# t)$. $\square$

Now we will discuss the question of relations beetween the left coquotient with respect to meet and the lattice operations of $\mathbb{PR}$.

Proposition 2.9. *(The left distributivity of left coquotient $r \,{}^{\wedge}\!/_{\#}\, s$ relative to join) Let $s \in \mathbb{PR}$. Then for every family of preradicals $\{ r_\alpha | \, \alpha \in \mathfrak{A} \}$ the following relation holds:*

$$\left(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right) {}^{\wedge}\!/_{\#}\, s = \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s).$$

Proof. $(\leq)$ By definition of left coquotient we have $r_\alpha \leq (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s$ for every $\alpha \in \mathfrak{A}$. Then $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \leq \bigvee_{\alpha \in \mathfrak{A}} [(r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s\,]$. From the distributivity of coproduct of preradicals relative to join it follows that $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \leq \left[\bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \right] \# s$. Using Proposition 2.5 we obtain $\left(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right) {}^{\wedge}\!/_{\#}\, s \leq \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s)$.

$(\geq)$ From Lemma 2.1 we have $\left(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right) {}^{\wedge}\!/_{\#}\, s = \wedge \left\{ t_\beta \in \mathbb{PR} \,|\; t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right\}$ and $\bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) = \bigvee_{\alpha \in \mathfrak{A}} \left(\bigwedge_{r'_\gamma \# s \geq r_\alpha} r'_\gamma \right)$.

Let $t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha$. Since $\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \geq r_\alpha$ for every $\alpha \in \mathfrak{A}$ we have $t_\beta \# s \geq r_\alpha$, so each preradical t_β is one of preradicals r'_γ. This implies the relation $\wedge \left\{ t_\beta \in \mathbb{PR} \,|\; t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right\} \geq \wedge \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \geq r_\alpha \}$ for every $\alpha \in \mathfrak{A}$, therefore $\wedge \left\{ t_\beta \in \mathbb{PR} \,|\; t_\beta \# s \geq \bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right\} \geq \bigvee_{\alpha \in \mathfrak{A}} (\wedge \{ r'_\gamma \in \mathbb{PR} \,|\, r'_\gamma \# s \geq r_\alpha \})$, which means that $\left(\bigvee_{\alpha \in \mathfrak{A}} r_\alpha \right) {}^{\wedge}\!/_{\#}\, s \geq \bigvee_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s)$. $\square$

Proposition 2.10. *In the class $\mathbb{PR}$ the following relations are true:*

1) $\left(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \right) {}^{\wedge}\!/_{\#}\, s \leq \bigwedge_{\alpha \in \mathfrak{A}} (r_\alpha \,{}^{\wedge}\!/_{\#}\, s)$;

2) $r \,{}^{\wedge}\!/_{\#} \left(\bigwedge_{\alpha \in \mathfrak{A}} s_\alpha \right) \geq \bigvee_{\alpha \in \mathfrak{A}} (r \,{}^{\wedge}\!/_{\#}\, s_\alpha)$;

3) $r \,{}^{\wedge}\!/_{\#} \left(\bigvee_{\alpha \in \mathfrak{A}} s_\alpha \right) \leq \bigwedge_{\alpha \in \mathfrak{A}} (r \,{}^{\wedge}\!/_{\#}\, s_\alpha)$.

Proof. 1) By the definition of left coquotient we have $r_\alpha \leq (r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s$ for every $\alpha \in \mathfrak{A}$, therefore $\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha \leq \bigwedge_{\alpha \in \mathfrak{A}} [(r_\alpha \,{}^{\wedge}\!/_{\#}\, s) \# s]$. From the distributivity of coproduct of preradicals relative to meet it follows that $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha \leq \left[\bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \,{}^{\wedge}\!\!/_{\#}\, s)\right] \# \, s$ and using Proposition 2.5 we obtain $\left(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\right) {}^{\wedge}\!\!/_{\#}\, s \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r_\alpha \,{}^{\wedge}\!\!/_{\#}\, s)$.

2) For every $\alpha \in \mathfrak{A}$ we have $\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha \leq s_\alpha$. From the antimonotony in the denominator of left coquotient it follows that $r \,{}^{\wedge}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \,{}^{\wedge}\!\!/_{\#} \left(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \geq \bigvee\limits_{\alpha\in\mathfrak{A}} (r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha)$.

3) For every $\alpha \in \mathfrak{A}$ we have $\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha \geq s_\alpha$. From the antimonotony in the denominator of left coquotient it follows that $r \,{}^{\wedge}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha$ for all $\alpha \in \mathfrak{A}$, therefore $r \,{}^{\wedge}\!\!/_{\#} \left(\bigvee\limits_{\alpha\in\mathfrak{A}} s_\alpha\right) \leq \bigwedge\limits_{\alpha\in\mathfrak{A}} (r \,{}^{\wedge}\!\!/_{\#}\, s_\alpha)$. □

3 The left coquotient $r \,{}^{\wedge}\!\!/_{\#}\, s$ in particular cases

In this section we study some particular cases of left coquotient with respect to meet, its relations with special constructions in large complete lattice $\mathbb{PR}$ and the connection with some types of preradicals (coprime, $\vee$-coprime, coirreducible), as well as the arrangement (relative position) of preradicals obtained by the studied operation.

Proposition 3.1. *For every preradicals $r, s \in \mathbb{PR}$ the following conditions are equivalent:*

1) $r \leq s$;

2) $r \,{}^{\wedge}\!\!/_{\#}\, s = 0$.

Proof. 1) $\Rightarrow$ 2) Let $r \leq s$. So $r \leq 0 \# s$ and from Proposition 2.5 we obtain $r \,{}^{\wedge}\!\!/_{\#}\, s \leq 0$, therefore $r \,{}^{\wedge}\!\!/_{\#}\, s = 0$.

2) $\Rightarrow$ 1) Let $r \,{}^{\wedge}\!\!/_{\#}\, s = 0$. By definition of left coquotient we have $(r \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq r$, so $0 \# s \geq r$, i.e $s \geq r$. □

Proposition 3.2. *Let $r, s \in \mathbb{PR}$. Then:*

1) $1 \,{}^{\wedge}\!\!/_{\#}\, s = t\,(s)$ *(see Def. 1.1);*

2) $r \,{}^{\wedge}\!\!/_{\#}\, 0 = r$.

Proof. From the definition of left coquotient we have:

1) $1 \,{}^{\wedge}\!\!/_{\#}\, s = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s \geq 1 \} = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# s = 1 \} = t\,(s)$;

2) $r \,{}^{\wedge}\!\!/_{\#}\, 0 = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \# 0 \geq r \} = \wedge \{ r_\alpha \in \mathbb{PR} \mid r_\alpha \geq r \} = r$. □

From Propositions 3.1 and 3.2 such particular cases follow:

(1) $0 \,{}^{\wedge}\!\!/_{\#}\, 0 = 0$; (2) $r \,{}^{\wedge}\!\!/_{\#}\, r = 0$ for every $r \in \mathbb{PR}$;

(3) $s \,{}^{\wedge}\!\!/_{\#}\, 1 = 0$ for every $s \in \mathbb{PR}$; (4) $1 \,{}^{\wedge}\!\!/_{\#}\, 1 = t\,(1) = 0$.

As in Proposition 3.1 $(r \,{}^{\wedge}\!\!/_{\#}\, r) \# r = 0 \# r = r$ for every $r \in \mathbb{PR}$.

Moreover, the distributivity of coproduct of preradicals relative to meet implies $t\,(s) \# s = \left(\underset{r_\alpha \# s = 1}{\wedge} r_\alpha \right) \# s = \underset{r_\alpha \# s = 1}{\wedge} (r_\alpha \# s) = 1$ for every $s \in \mathbb{PR}$.

Now we will indicate the relations between the totalizer $t\,(r)$ of preradical r and such constructions in the large complete lattice $\mathbb{PR}$ as pseudocomplement and supplement (see Def. 1.2, Def. 1.3).

Proposition 3.3. *For every preradical $s \in \mathbb{PR}$ we have $t\,(s) \geq s^{\perp}$.*

Proof. By definition $t\,(s) = \wedge \{ r_\alpha \mid r_\alpha \# s = 1 \}$. The pseudocomplement $s^{\perp}$ of preradical s by definition has the property $s \wedge s^{\perp} = 0$. Since $s \cdot s^{\perp} \leq s \wedge s^{\perp} = 0$, we obtain $s \cdot s^{\perp} = 0$. We have that $t\,(s) \# s = 1$, so $s^{\perp} = 1 \cdot s^{\perp} = (t\,(s) \# s) \cdot s^{\perp}$. From Lemma 1.4 $(t\,(s) \# s) \cdot s^{\perp} \leq \big(t\,(s) \cdot s^{\perp}\big) \# \big(s \cdot s^{\perp}\big) = \big(t\,(s) \cdot s^{\perp}\big) \# 0 = t\,(s) \cdot s^{\perp}$. Therefore $s^{\perp} \leq t\,(s) \cdot s^{\perp}$, but $t\,(s) \cdot s^{\perp} \leq t\,(s)$, so $s^{\perp} \leq t\,(s)$. □

Moreover, we have $s^{\perp} \leq t\,(s) \cdot s^{\perp}$, but $s^{\perp} \geq t\,(s) \cdot s^{\perp}$, so $s^{\perp} = t\,(s) \cdot s^{\perp}$.

Proposition 3.4. *Let $s \in \mathbb{PR}$ and s have the supplement s^*. Then $t\,(s) \leq s^*$.*

Proof. By definition $t\,(s) = \wedge \{ r_\alpha \mid r_\alpha \# s = 1 \}$. The supplement s^* of s from the definition has the property $s \vee s^* = 1$. Since $s^* \# s \geq s^* \vee s = s \vee s^* = 1$, we obtain $s^* \# s = 1$. So s^* is one of preradicals r_α, therefore $s^* \geq \wedge \{ r_\alpha \mid r_\alpha \# s = 1 \}$, i.e. $s^* \geq t\,(s)$. □

Moreover, from Proposition 2.3 $r \,{}^{\wedge}\!\!/_{\#}\, s \leq 1 \,{}^{\wedge}\!\!/_{\#}\, s = t\,(s)$, therefore $r \,{}^{\wedge}\!\!/_{\#}\, s \leq s^*$.

The next two statements show when the cancellation properties for left coquotient hold (see Proposition 2.6).

Proposition 3.5. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \# s) \,{}^{\wedge}\!\!/_{\#}\, s$;

2) $r = t \,{}^{\wedge}\!\!/_{\#}\, s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) If $r = (r \# s) \,{}^{\wedge}\!\!/_{\#}\, s$, then $r = t \,{}^{\wedge}\!\!/_{\#}\, s$ with $t = r \# s$.

2) $\Rightarrow$ 1) Let $r = t \,{}^{\wedge}\!\!/_{\#}\, s$ for some preradical t. By definition of left coquotient $(t \,{}^{\wedge}\!\!/_{\#}\, s) \# s \geq t$. From Proposition 2.3 we obtain $[(t \,{}^{\wedge}\!\!/_{\#}\, s) \# s] \,{}^{\wedge}\!\!/_{\#}\, s \geq t \,{}^{\wedge}\!\!/_{\#}\, s$. But from Proposition 2.6 $[(t \,{}^{\wedge}\!\!/_{\#}\, s) \# s] \,{}^{\wedge}\!\!/_{\#}\, s \leq t \,{}^{\wedge}\!\!/_{\#}\, s$, therefore we have $[(t \,{}^{\wedge}\!\!/_{\#}\, s) \# s] \,{}^{\wedge}\!\!/_{\#}\, s = t \,{}^{\wedge}\!\!/_{\#}\, s$. Since $t \,{}^{\wedge}\!\!/_{\#}\, s = r$, we obtain $(r \# s) \,{}^{\wedge}\!\!/_{\#}\, s = r$. □

Proposition 3.6. *Let $r, s \in \mathbb{PR}$. The following conditions are equivalent:*

1) $r = (r \,{}^{\wedge}\!\!/_{\#}\, s) \# s$;

2) $r = t \# s$ *for some preradical* $t \in \mathbb{PR}$.

Proof. 1) $\Rightarrow$ 2) If $r = (r \,{}^{\wedge}\!/_{\#}\, s) \# s$, then $r = t \# s$ with $t = r \,{}^{\wedge}\!/_{\#}\, s$.

2) $\Rightarrow$ 1) Let $r = t \# s$ for some preradical t. By Proposition 2.6 $(t \# s) \,{}^{\wedge}\!/_{\#}\, s \leq t$. From the monotony of coproduct it follows that $[(t \# s) \,{}^{\wedge}\!/_{\#}\, s] \# s \leq t \# s$. But from the definition of left coquotient $[(t \# s) \,{}^{\wedge}\!/_{\#}\, s] \# s \geq t \# s$, therefore $[(t \# s) \,{}^{\wedge}\!/_{\#}\, s] \# s = t \# s$. Since $t \# s = r$, we have $(r \,{}^{\wedge}\!/_{\#}\, s) \# s = r$. □

Now we will study the behaviour of the left coquotient $r \,{}^{\wedge}\!/_{\#}\, s$ in the cases of such types of preradicals as coprime, $\vee$-coprime and coirreducible.

Proposition 3.7. *The preradical r is coprime if and only if for every preradical s we have $r \,{}^{\wedge}\!/_{\#}\, s = 0$ or $r \,{}^{\wedge}\!/_{\#}\, s = r$.*

Proof. ($\Rightarrow$) Let $r \neq 0$. By definition $(r \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r$ and if r is coprime, then we have $r \,{}^{\wedge}\!/_{\#}\, s \geq r$ or $s \geq r$. If $r \,{}^{\wedge}\!/_{\#}\, s \geq r$, then since by Lemma 2.2 $r \,{}^{\wedge}\!/_{\#}\, s \leq r$, it follows that $r \,{}^{\wedge}\!/_{\#}\, s = r$. If $s \geq r$, then from Proposition 3.1 we have $r \,{}^{\wedge}\!/_{\#}\, s = 0$.

($\Leftarrow$) Let $t_1 \# t_2 \geq r$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. From Proposition 2.5 we obtain $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, t_2$. For the preradical t_2 from the condition of this proposition we have $r \,{}^{\wedge}\!/_{\#}\, t_2 = 0$ or $r \,{}^{\wedge}\!/_{\#}\, t_2 = r$. If $r \,{}^{\wedge}\!/_{\#}\, t_2 = 0$, then from Proposition 3.1 it follows that $t_2 \geq r$. If $r \,{}^{\wedge}\!/_{\#}\, t_2 = r$, then $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, t_2 = r$. So for every $t_1, t_2 \in \mathbb{PR}$ with $t_1 \# t_2 \geq r$ we have $t_1 \geq r$ or $t_2 \geq r$, which means that the preradical r is coprime. □

Proposition 3.8. *If the preradical r is $\vee$-coprime, then the coquotient $r \,{}^{\wedge}\!/_{\#}\, s$ is $\vee$-coprime for every $s \in \mathbb{PR}$.*

Proof. Suppose that $t_1 \vee t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$, for some $t_1, t_2 \in \mathbb{PR}$. Then from Proposition 2.5 we obtain $(t_1 \vee t_2) \# s \geq r$. From the distributivity of coproduct of preradicals relative to join we have $(t_1 \# s) \vee (t_2 \# s) \geq r$. If r is $\vee$-coprime, then $t_1 \# s \geq r$ or $t_2 \# s \geq r$. From Proposition 2.5 we obtain that $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, s$ or $t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$. So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \vee t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$ we have $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, s$ or $t_2 \geq r \,{}^{\wedge}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is $\vee$-coprime. □

Proposition 3.9. *Let $r, s \in \mathbb{PR}$ and $r = t \# s$ for some preradical $t \in \mathbb{PR}$. If the preradical r is coirreducible, then the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is coirreducible.*

Proof. Let $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\#}\, s$ for some preradicals $t_1, t_2 \in \mathbb{PR}$. If $r = t \# s$ for some preradical t, then by Proposition 3.6 $r = (r \,{}^{\wedge}\!/_{\#}\, s) \# s$, so $r = (t_1 \vee t_2) \# s$. From the distributivity of coproduct of preradicals relative to join $r = (t_1 \# s) \vee (t_2 \# s)$. If r is coirreducible, then $t_1 \# s = r$ or $t_2 \# s = r$.

If $t_1 \# s = r$, then from Proposition 2.5 we have $t_1 \geq r \,{}^{\wedge}\!/_{\#}\, s$. But $t_1 \leq r \,{}^{\wedge}\!/_{\#}\, s$, because $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\#}\, s$, therefore $t_1 = r \,{}^{\wedge}\!/_{\#}\, s$.

If $t_2 \# s = r$, then similarly we obtain $t_2 = r \,{}^{\wedge}\!/_{\#}\, s$.

So for every preradicals $t_1, t_2 \in \mathbb{PR}$ with $t_1 \vee t_2 = r \,{}^{\wedge}\!/_{\#}\, s$ we have $t_1 = r \,{}^{\wedge}\!/_{\#}\, s$ or $t_2 = r \,{}^{\wedge}\!/_{\#}\, s$, which means that the preradical $r \,{}^{\wedge}\!/_{\#}\, s$ is coirreducible. □

The operation of left coquotient with respect to meet implies some order relations between the associated preradicals. To see that we firstly prove

Proposition 3.10. *For every* $r, s, t \in \mathbb{PR}$ *the following relations are true:*

1) $r \,{}^{\wedge}\!/_{\#}\, s = (r \vee s) \,{}^{\wedge}\!/_{\#}\, s$;

2) $(r \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r \vee s$.

Proof. 1) From Proposition 2.9 we have that $(r \vee s) \,{}^{\wedge}\!/_{\#}\, s = (r \,{}^{\wedge}\!/_{\#}\, s) \vee (s \,{}^{\wedge}\!/_{\#}\, s)$, but $s \,{}^{\wedge}\!/_{\#}\, s = 0$, so $(r \vee s) \,{}^{\wedge}\!/_{\#}\, s = (r \,{}^{\wedge}\!/_{\#}\, s) \vee 0 = r \,{}^{\wedge}\!/_{\#}\, s$.

Moreover, since $r \# s \geq r \vee s$ from Proposition 2.3 we obtain

$$(r \# s) \,{}^{\wedge}\!/_{\#}\, s \geq (r \vee s) \,{}^{\wedge}\!/_{\#}\, s = r \,{}^{\wedge}\!/_{\#}\, s.$$

2) By 1) we have $r \,{}^{\wedge}\!/_{\#}\, s = (r \vee s) \,{}^{\wedge}\!/_{\#}\, s$ and so $(r \,{}^{\wedge}\!/_{\#}\, s) \# s = ((r \vee s) \,{}^{\wedge}\!/_{\#}\, s) \# s$. From the definition of left coquotient we have $((r \vee s) \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r \vee s$, therefore $(r \,{}^{\wedge}\!/_{\#}\, s) \# s \geq r \vee s$. □

Corollary 3.11. 1) *For every preradicals* $r, s \in \mathbb{PR}$ *the following relations hold:*

$$r \,{}^{\wedge}\!/_{\#}\, s \leq (r \# s) \,{}^{\wedge}\!/_{\#}\, s \leq r \leq r \vee s \leq (r \,{}^{\wedge}\!/_{\#} s) \# s \leq r \# s;$$

2) *If* r *is cohereditary, then*

$$r \,{}^{\wedge}\!/_{\#}\, s = (r \# s) \,{}^{\wedge}\!/_{\#}\, s \leq r \leq r \vee s = (r \,{}^{\wedge}\!/_{\#} s) \# s = r \# s$$

for every $s \in \mathbb{PR}$. □

We remark that the operations of left quotient with respect to join and left coquotient with respect to meet are complete in the sense of existence for any two preradicals.

In conclusion, we can say that in this work is introduced and studied a new (complete) operation (*left coquotient with respect to meet*) in the class of preradicals $\mathbb{PR}$ of R-Mod, which is dual the previous operation (*left quotient with respect to join*) and possesses similar properties. The indicated facts dualise the results of paper [2]. In the particular case of pretorsions as corrolaries we obtain a series of results of J. S. Golan [1], as is indicated in [3].

References

[1] Golan J. S. *Linear topologies on a ring.* Logman Sci. Techn., New York, 1987.

[2] Jardan I. *On the inverse operations in the class of preradicals of a module category, I.* Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2017, No. 1(83), 57–66.

[3] Jardan I. *On the left coquotient with respect to meet for pretorsions in modules.* The 4^{th} Conference of Mathematical Society of the Republic of Moldova, Chisinau, Moldova, 2017, 95-98.

[4] Bican L., Kepka T., Nemec P. *Rings, modules and preradicals.* Marcel Dekker, New York, 1982.

[5] Kashu A. I. *Radicals and torsions in modules.* Kishinev, Shtiintsa, 1983 (in Russian).

[6] Raggi F., Rios J., Rincon H., Fernandes-Alonso R., Signoret C. *The lattice structure of preradicals.* Commun. Algebra, 2002, No. 30(3), 1533–1544.

[7] Raggi F., Rios J., Rincon H., Fernandes-Alonso R., Signoret C. *The lattice structure of preradicals II.* J. of Algebra and its Applications, 2002, Vol. 1, No. 2, 201–214.

[8] Kashu A. I. *On some operations in the lattice of submodules determined by preradicals*. Bul. Acad. Ştiinţe Repub. Moldova, Mat., 2011, No. 2(66), 5–16.

[9] Raggi F., Rios J., Wisbauer R. *Coprime preradicals and modules*. Journal of Pure and Applied Algebra, 2005, **200**, 51–69.

[10] Kashu A. I. *On inverse operations in the lattices of submodules*. Algebra and Discrete Mathematics, 2012, Vol. 13, No. 2, 273–288.

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