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Invariant conditions of stability of unperturbed motion governed by some differential systems in the plane

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Abstract. Center-affine invariant conditions of the stability of unperturbed motion were determined for differential systems in the plane with polynomial nonlinearities in non-critical cases and for differential systems in the plane with polynomial nonlinearities up to the fourth degree inclusive in critical cases.

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Introduction

Problems which require a general formulation of stability not only of equilibrium but also of motion arose in science and technics in the middle of XIX-th century.

Lyapunov (1857-1918) published his PhD thesis concerning the stability of motion in 1892, and it was translated into French and published in France in 1907. According to the French version, this work was reprinted in Russian, with some additions, in his collection of works [1] in 1956. The mentioned work contains many fruitful ideas and results of great importance. All the history related to the theory on stability of motion is considered to be divided into periods before and after Lyapunov.

First of all, A. M. Lyapunov gave a strict definition of the stability of motion, which was so successful that all scientists took it as fundamental one for their researches.

A lot of papers were written in the field of stability of motion. The universal scientific literature concerning the stability of motion contains thousands of papers, including hundreds of monographs and textbooks of many authors. This literature is rich in the development of this theory, as well as in its applications in practice.

Note that many problems on stability treated in these works are governed by two-dimensional (or multidimensional) autonomous polynomial differential systems. Methods of the theory of invariants for such systems were elaborated in the school of differential equations from Chişinău. Moreover, the theory of Lie algebras and Sibirsky graded algebras with applications in the qualitative theory of these equations [2–7] there were developed.

The stability of unperturbed motions using the theory of algebras, of invariants and of Lie algebras was studied for the first time in [8]. In this paper, the similar investigations are done for two-dimensional differential systems with polynomial nonlinearities.

1 Definition of stability of unperturbed motion and of critical system

We consider the two-dimensional differential system with polynomial nonlinearities of perturbed motion (see, for example, [1] or [9]) of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + \sum_{i=1}^{l} a^j_{\alpha_1\alpha_2\ldots\alpha_{m_i}} x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_{m_i}} \quad (j, \alpha, \alpha_1, \ldots, \alpha_{m_i} = 1, 2; l < \infty), \quad (1)$$

where $a^j_{\alpha_1\alpha_2\ldots\alpha_{m_i}}$ is a symmetric tensor in lower indices in which the total convolution is done and $\Gamma = \{m_1, m_2, \ldots, m_l\}$ ($m_i \geq 2$) is a finite set of distinct natural numbers. Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$.

The system of the first approximation ([1], [9])

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha \quad (j, \alpha = 1, 2) \quad (2)$$

plays an important role in studying differential systems (1). As it follows from [1] (or [9]), to unperturbed motion of system (1) the zero values of variables $x^j(t)$ ($j = 1, 2$) correspond. Taking into account this fact, we have the following *definition of stability by Lyapunov [9]*:

If for any small positive value ε, however small, one can find a positive number δ such that for all perturbations $x^j(t_0)$ satisfying the condition

$$\sum_{j=1}^{2} (x^j(t_0))^2 \leq \delta, \quad (3)$$

the inequality

$$\sum_{j=1}^{2} (x^j(t))^2 < \varepsilon$$

is valid for any $t \geq t_0$, then the unperturbed motion $x^j = 0$ ($j = \overline{1, 2}$) is called stable, otherwise it is called unstable.

If the unperturbed motion is stable and the number δ can be found however small such that for any perturbed motions satisfying (3) the condition

$$\lim_{t \to \infty} \sum_{j=1}^{2} (x^j(t))^2 = 0,$$

is valid, then the unperturbed motion is called *asymptotically stable.*

Inspired by the work [1] we have

Definition 1. The differential system (1) with polynomial nonlinearities will be called *a critical system of Lyapunov type* if the characteristic equation of the system of the first approximation (2) has one zero root and all other roots have negative real parts. When the real parts of the roots of the characteristic equation are different from zero, the system (1) will be called *non-critical.*

First, we will examine the non-critical case.

Lemma 1. *The characteristic equation of system* (1) *and* (2) *is*

$$\varrho^2 + L_{1,2}\varrho + L_{2,2} = 0, \tag{4}$$

where the coefficients in (4) *are center-affine invariants [2] and have the form*

$$L_{1,2} = -I_1, \quad L_{2,2} = \frac{1}{2}(I_1^2 - I_2) \tag{5}$$

with

$$I_1 = a^\alpha_\alpha, \ \ I_2 = a^\alpha_\beta a^\beta_\alpha. \tag{6}$$

By means of the Lyapunov theorems on stability of unperturbed motion in the first approximation (2), the Hurwitz theorem on the signs of the roots of an algebraic equation (see, for example, [9]) and using Lemma 1 we have

Theorem 1. *Assume that the center-affine invariants* (5) *of system* (1) *satisfy the inequalities* $L_{1,2} > 0, \ \ L_{2,2} > 0$. *Then the unperturbed motion* $x^1 = x^2 = 0$ *of this system is asymptotically stable.*

Theorem 2. *If at least one of the center-affine invariant expressions* (5) *of system* (1) *is negative, then the unperturbed motion* $x^1 = x^2 = 0$ *of this system is unstable.*

2 Canonical form of a critical system of Lyapunov type

Remark 1. In the following, we will study critical systems of Lyapunov type in the first case, and such systems will be called *critical systems* or *critical systems of Lyapunov type.*

Lemma 2. *The characteristic equation of system* (2) *(and therefore of system* (1)*) has one zero root and the other ones real and negative if and only if the following invariant conditions*

$$I_1^2 - I_2 = 0, \ \ I_1 < 0 \tag{7}$$

hold, where I_1 *and* I_2 *are from* (6).

The proof of Lemma 2 follows from the fact that the characteristic equation of system (2) and therefore of (1) has the form (4)–(5).

From [1] it follows

Lemma 3. *Let for system* (2) *(for* (1)*) the invariant conditions* (7) *hold. Then the system* (2) *by a center-affine transformation can be brought to the form*

$$\frac{dx^1}{dt} = 0, \ \ \frac{dx^2}{dt} = a^2_\alpha x^\alpha \ \ (\alpha = \overline{1,2}) \tag{8}$$

and, therefore, the system (1) *can be written in the form*

$$\frac{dx^1}{dt} = \sum_{i=1}^{l} a^1_{\alpha_1\alpha_2\ldots\alpha_{m_i}} x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_{m_i}},$$
$$\frac{dx^2}{dt} = a^2_\alpha x^\alpha + \sum_{i=1}^{l} a^2_{\alpha_1\alpha_2\ldots\alpha_{m_i}} x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_{m_i}} \ (\alpha,\alpha_1,\ldots,\alpha_{m_i} = \overline{1,2};\ l<\infty). \tag{9}$$

Remark 2. The system (9) is called the canonical form of a critical system of Lyapunov type (1), where the first equation from (9) is called the critical equation and the second one – the non-critical equation.

For the case examined in this paper the Lyapunov's Theorem [1, §32] can be written in the following form:

Theorem 3. *Let the characteristic equation of the matrix of linear part of differential system with polynomial nonlinearities have one zero root and other roots have negative real parts. Assume that the differential system of the perturbed motion* (1) *was brought to the form* (9) *and consider the equation*

$$a^2_\alpha x^\alpha + \sum_{i=1}^{l} a^2_{\alpha_1\alpha_2\ldots\alpha_{m_i}} x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_{m_i}} = 0 \ \ (\alpha,\alpha_1,\alpha_2,\ldots,\alpha_{m_i} = \overline{1,2};\ l<\infty) \tag{10}$$

from which we determine the variable x^2 as a holomorphic function of the variable x^1, vanishing for $x^1 = 0$ (such determination of x^2 is always possible and is unique). Substitute the determined values into the polynomial

$$\sum_{i=1}^{l} a^1_{\alpha_1\alpha_2\ldots\alpha_{m_i}} x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_{m_i}} \ \ (\alpha_1,\alpha_2,\ldots,\alpha_{m_i} = 1,2;\ l<\infty).$$

If the obtained result is not identically zero, then we can develop it in an increasing powers series of x^1. When the lowest power of x^1 in this development is even, then the unperturbed motion is unstable. When the lowest power of x^1 is odd, then the unperturbed motion depends on the sign of the coefficient of x^1. The unperturbed motion will be unstable when this coefficient is positive and will be stable when the coefficient is negative. In the last case, any perturbed motion that corresponds to small enough perturbation will approach asymptotically the unperturbed motion.

If the obtained result is identically zero, then there exists a continuous series of stabilized motions to which the examined unperturbed motion belongs. All the motions of this series, close enough to the unperturbed motions, including the last one, will be stable. In this case, for small enough perturbations, any perturbed motion will tend asymptotically to one of the stabilized motions of the mention series.

3 Center-affine invariant conditions of stability of unperturbed motion for the system with quadratic nonlinearities

We examine the differential system with quadratic nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \ \ (j, \alpha, \beta = 1, 2), \tag{11}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices in which the total convolution is done.

It was shown in [4] that the set of unimodular comitants and invariants of the system (1) consists of some graded algebras, which in [7] were called the Sibirsky algebras. For system (11) these algebras were denoted in [4] by $S_{1,2}$ – the Sibirsky algebras of comitants and $SI_{1,2}$ – the Sibirsky algebras of invariants.

It was shown in [4] that the set of generators of these algebras (which is finite) consists of polynomial bases of the homogeneous center-affine comitants and invariants.

Based on this and on the polynomial bases of the center-affine comitants and invariants of system (11) given in [2], we can write the Sibirsky algebra in the form

$$S_{1,2} =< I_1, I_2, \ldots, I_{16}, K_1, K_2, \ldots, K_{20} \,|\, f_1, f_2, \ldots, f_{27} >$$

and

$$SI_{1,2} =< I_1, I_2, \ldots, I_{16} \,|\, f_1, f_2, \ldots, f_9 >,$$

where I_r and K_s are the invariants and the comitants of these algebras, and f_j are their syzygies.

Later on, we will use the following generators of the Sibirsky algebras of system (11), for which their tensorial forms from [2] are written as follows:

$$\begin{gathered} I_1 = a^\alpha_\alpha, \ \ I_2 = a^\alpha_\beta a^\beta_\alpha, \ \ I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta} \varepsilon^{pq}, \ \ K_1 = a^\alpha_{\alpha\beta} x^\beta, \ \ K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq}, \\ K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma} x^\gamma, \ \ K_4 = a^\alpha_\gamma a^\beta_{\alpha\beta} x^\gamma, \ \ K_5 = a^p_{\alpha\beta} x^\alpha x^\beta x^q \varepsilon_{pq}, \ \ K_7 = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} x^\gamma x^\delta, \\ K_8 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta} x^\delta, \ \ K_{11} = a^p_\alpha a^\alpha_{\beta\gamma} x^\beta x^\gamma x^q \varepsilon_{pq}, \ \ K_{12} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} x^\delta x^\mu, \\ K_{13} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma_{\delta\mu} x^\delta x^\mu, \end{gathered} \tag{12}$$

where $\varepsilon^{pq}(\varepsilon_{pq})$ is the unit bivector with coordinates $\varepsilon^{11} = \varepsilon^{22} = 0, \varepsilon^{12} = -\varepsilon^{21} = 1$ $(\varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = -\varepsilon_{21} = 1)$.

Suppose the system (11) is critical of Lyapunov type. Then by Lemma 3 it can be brought to the canonical form (9)

$$\frac{dx^1}{dt} = a^1_{\alpha\beta} x^\alpha x^\beta, \ \ \frac{dx^2}{dt} = a^2_\alpha x^\alpha + a^2_{\alpha\beta} x^\alpha x^\beta \ \ (\alpha, \beta = 1, 2). \tag{13}$$

According to Theorem 3, we examine the equation (10) provided by non-critical equation of (13), which in the expanded form looks as

$$a^2_1 x^1 + a^2_2 x^2 + a^2_{11}(x^1)^2 + 2a^2_{12} x^1 x^2 + a^2_{22}(x^2)^2 = 0. \tag{14}$$

In this case, under the conditions (5)–(6) and the inequality from (7) we have

$$I_1 = a_2^2 < 0. \tag{15}$$

Then from (14) we can write

$$x^2 = -\frac{a_1^2}{a_2^2}x^1 - \frac{a_{11}^2}{a_2^2}(x^1)^2 - \frac{2a_{12}^2}{a_2^2}x^1x^2 - \frac{a_{22}^2}{a_2^2}(x^2)^2. \tag{16}$$

By Theorem 3, we seek x^2 as a holomorphic function of x^1. Then we can write

$$x^2 = -\frac{a_1^2}{a_2^2}x^1 + B_2(x^1)^2 + B_3(x^1)^3 + B_4(x^1)^4 + \cdots \tag{17}$$

Substituting (17) into (16) we get

$$-\frac{a_1^2}{a_2^2}x^1 + B_2(x^1)^2 + B_3(x^1)^3 + \cdots = -\frac{a_1^2}{a_2^2}x^1 - \frac{a_{11}^2}{a_2^2}(x^1)^2 - \frac{2a_{12}^2}{a_2^2}x^1[-\frac{a_1^2}{a_2^2}x^1 +$$
$$+ B_2(x^1)^2 + B_3(x^1)^3 + \cdots] - \frac{a_{22}^2}{a_2^2}[-\frac{a_1^2}{a_2^2}x^1 + B_2(x^1)^2 + B_3(x^1)^3 + \cdots]^2.$$

This implies that

$$B_2(x^1)^2 + B_3(x^1)^3 + B_4(x^1)^4 + \cdots = [-\frac{a_{11}^2}{a_2^2} + \frac{2a_1^2a_{12}^2}{(a_2^2)^2} - \frac{(a_1^2)^2a_{22}^2}{(a_2^2)^3}](x^1)^2 +$$
$$+[-\frac{2a_{12}^2}{a_2^2}B_2 + \frac{2a_1^2a_{22}^2}{(a_2^2)^2}B_2](x^1)^3 + [-\frac{2a_{12}^2}{a_2^2}B_3 - \frac{a_{22}^2}{a_2^2}B_2^2 + 2\frac{a_1^2a_{22}^2}{(a_2^2)^2}B_3](x^1)^4 + \cdots$$

and we obtain

$$B_2 = \frac{1}{(a_2^2)^3}[-(a_2^2)^2a_{11}^2 + 2a_1^2a_2^2a_{12}^2 - (a_1^2)^2a_{22}^2],\ B_3 = \frac{2}{(a_2^2)^2}(-a_2^2a_{12}^2 + a_1^2a_{22}^2)B_2,$$
$$B_4 = \frac{1}{(a_2^2)^2}[-a_2^2a_{22}^2B_2^2 + 2(a_1^2a_{22}^2 - a_2^2a_{12}^2)B_3], \ldots \tag{18}$$

Substituting (17) into the right-hand side of the critical differential equation (13) we have

$$a_{11}^1(x^1)^2 + 2a_{12}^1x^1x^2 + a_{22}^1(x^2)^2 = A_2(x^1)^2 + A_3(x^1)^3 + A_4(x^1)^4 + \cdots$$

or in the expanded form we get

$$a_{11}^1(x^1)^2 + 2a_{12}^1x^1[-\frac{a_1^2}{a_2^2}x^1 + B_2(x^1)^2 + B_3(x^1)^3 + \cdots] +$$

$$+a_{22}^1[-\frac{a_1^2}{a_2^2}x^1 + B_2(x^1)^2 + B_3(x^1)^3 + \cdots]^2 = A_2(x^1)^2 + A_3(x^1)^3 + A_4(x^1)^4 + \cdots .$$

This implies that

$$A_2 = \frac{1}{(a_2^2)^2}[(a_2^2)^2 a_{11}^1 - 2a_1^2 a_2^2 a_{12}^1 + (a_1^2)^2 a_{22}^1],$$
$$A_3 = \frac{2}{a_2^2}(a_2^2 a_{12}^1 - a_1^2 a_{22}^1)B_2,\ A_4 = \frac{2}{a_2^2}(a_2^2 a_{12}^1 - a_1^2 a_{22}^1)B_3 + a_{22}^1 B_2^2, \dots \tag{19}$$

By Theorem 3, to determine the stability of the unperturbed motion described by system (13), it is necessary to study the expressions (19).

Let us introduce the following notations

$$P = (a_2^2)^2 a_{11}^1 - 2a_1^2 a_2^2 a_{12}^1 + (a_1^2)^2 a_{22}^1,\ \ Q = (a_2^2)^2 a_{11}^2 - 2a_1^2 a_2^2 a_{12}^2 + (a_1^2)^2 a_{22}^2,$$
$$R = (a_2^2)^2 a_{11}^1 - (a_1^2)^2 a_{22}^1,\ \ S = a_1^2 a_{22}^1 - a_2^2 a_{12}^1 \tag{20}$$

and take into account that according to (15) we have $a_2^2 < 0$.

Next, we observe that the stability of the unperturbed motion can occur when A_2=0 from (19), i.e. when $P = 0$ from (20).

Assume in (18) that $B_2 = 0$, then (20) yields $Q = 0$. This implies that all $B_3, B_4, ...$ are equal to zero. From this it follows that all the coefficients $A_3, A_4, ...$ vanish and therefore the stability of the unperturbed motion holds.

Suppose $B_2 \neq 0$. If $S \neq 0$, then the stability of the unperturbed motion is determined by the sign of A_3 from (19). If in (20) $S = 0$, then $A_3 = 0$ and the coefficient A_4 from (19) is non-zero if $a_{22}^1 \neq 0$. Therefore, the stability is possible only if $a_{22}^1 = 0$. Observe that when $S = P = 0$ in (20), then $R = 0$. Hence, when $a_{22}^1 = 0$ the last two equations in (20) yield $a_{11}^1 = a_{12}^1 = 0$.

Taking into account the inequality (15) and Theorem 3, we obtain the following results for stability of the unperturbed motion determined by the system of perturbed motion (13).

Lemma 4. *The stability of the unperturbed motion described by system* (13) *under conditions* (7) *is characterized by one of the following six possible cases:*

I. $P \neq 0$, *then the unperturbed motion is unstable;*

II. $P = 0, QS > 0$, *then the unperturbed motion is unstable;*

III. $P = 0, QS < 0$, *then the unperturbed motion is stable;*

IV. $R = S = 0,\ a_{22}^1 Q \neq 0$, *then the unperturbed motion is unstable;*

V. $P = Q = 0$, *then the unperturbed motion is stable;*

VI. $a_{11}^1 = a_{12}^1 = a_{22}^1 = 0$, *then the unperturbed motion is stable.*

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, moreover, it is also asymptotically stable [10] in Case III. The expressions P, Q, R, S *are given in* (20).

Later on, we make use of the following expressions of the invariants and comitants of system (11) given in (12):

$$\begin{gathered}E_1 = I_1^2 K_1 - I_1(K_3 + K_4) + K_8,\\ E_2 = I_1^3(K_1^2 - K_7) + 2I_1^2(K_1K_4 - 2K_1K_3 - K_{13}) + 2I_1(I_5K_2 + 2K_3^2 - K_4^2) +\\ +4K_8(K_4 - K_3) + 2I_2K_{12},\ \ E_3 = I_2K_1 + I_1(K_4 - K_3) - K_8,\\ E_4 = I_1(K_{11} - K_1K_2) + K_2(K_4 - K_3),\ \ E_5 = K_{11} - I_1K_5.\end{gathered} \tag{21}$$

Lemma 5. *Suppose the first equality from* (7) *holds. Then the system* (11) *by a center-affine transformation can be brought to the form*

$$\frac{dx^1}{dt} = 0,\ \ \frac{dx^2}{dt} = a^2_\alpha x^\alpha + a^2_{\alpha\beta}x^\alpha x^\beta\ \ (\alpha, \beta = 1, 2) \tag{22}$$

if and only if the following condition

$$E_5 \equiv 0, \tag{23}$$

holds, where E_5 *is from* (21).

Proof. Suppose the first relation from (7) holds. This allows us to write for (11)

$$a^1_1 = ra^2_1,\quad a^1_2 = ra^2_2. \tag{24}$$

Denote by Δ_{ij} the minors of matrix of the coefficients from the right-hand sides of system (11), where i and j represent the number of columns of this matrix on which the minors are built. Then

$$E_5 = \Delta_{13}(x^1)^3 + (\Delta_{23} + 2\Delta_{14})(x^1)^2x^2 + (\Delta_{15} + 2\Delta_{24})x^1(x^2)^2 + \Delta_{25}(x^2)^3.$$

By means of this expressions and of conditions (23)–(24) we have

$$a^1_{11} = ra^2_{11},\quad a^1_{12} = ra^2_{12},\quad a^1_{22} = ra^2_{22}. \tag{25}$$

Taking into account (24) and (25), the center-affine transformation $\bar{x}^1 = x^1 - rx^2$, $\bar{x}^2 = x^2$ brings the system (11) to the form (15). Lemma 5 is proved. □

Theorem 4. *Let for differential system of the perturbed motion* (11) *the invariant conditions* (7) *be satisfied. Then the stability of the unperturbed motion in system* (11) *is described by one of the following six possible cases:*

I. $E_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $E_1 \equiv 0,\ \ E_2 > 0$, *then the unperturbed motion is unstable;*

III. $E_1 \equiv 0,\ \ E_2 < 0$, *then the unperturbed motion is stable;*

IV. $E_3 \equiv 0, E_4E_5 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $E_4 \equiv 0$, *then the unperturbed motion is stable;*

VI. $E_5 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, it is also asymptotically stable in Case III. The expressions E_i $(i = \overline{1, 5})$ *are given in* (21).

Proof. Observe that expressions (21), for system (13) under condition (15), are expressed by (20) as follows:

$$\begin{gathered} E_1 = Px^1, \ \ E_2 = 4S[Q(x^1)^2 - Px^1x^2], \\ E_3 = Rx^1 - 2a_2^2Sx^2, \ \ E_4 = -Q(x^1)^3 + P(x^1)^2x^2. \end{gathered} \tag{26}$$

Setting $E_3 \equiv 0$, then by means of the polynomials R and S from (20), we get for E_5 from (21) the expression $E_5 = -a_2^2 a_{22}^1 (\frac{a_1^2}{a_2^2}x^1 + x^2)^3$.

Using the last assertion, the expressions (22) and Lemmas 4 and 5, we get the Cases I-VI. We mention that the comitant E_2 from (21) is even with respect to x^1 and x^2 and has the weight equal to zero [2] in the Cases II and III. This ensures that any center-affine transformation cannot change the sign of E_2. Theorem 4 is proved. □

Remark 3. From Theorem 4, the conditions for Lyapunov's Example 2 [1, §32] are obtained setting $a_1^1 = a_2^1 = 0$, $a_1^2 = k$, $a_2^2 = -1$, $a_{11}^1 = a, a_{12}^1 = \frac{1}{2}b$, $a_{22}^1 = c$, $a_{11}^2 = l$, $a_{12}^2 = \frac{1}{2}m$, $a_{22}^2 = n$ and $x^1 = x$, $x^2 = y$.

4 Critical system of Lyapunov type with cubic nonlinearities

Let the differential system of perturbed motion with polynomial nonlinearities of the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= ex + fy + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \end{aligned} \tag{27}$$

where $c, d, e, f, p, q, r, s, t, u, v, w$ are arbitrary real coefficients.

Similar to the previous case, when the characteristic equation of (27) has one zero root and the other one is negative, i.e. the conditions (7) are satisfied, then system (27) by a center-affine transformation can be brought to its critical form

$$\begin{aligned} \frac{dx}{dt} &= px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \frac{dy}{dt} &= ex + fy + tx^3 + 3ux^2y + 3vxy^2 + wy^3. \end{aligned} \tag{28}$$

According to (10) we write the equation

$$ex + fy + tx^3 + 3ux^2y + 3vxy^2 + wy^3 = 0. \tag{29}$$

By (6)–(7) we have for system (28) that $I_1 = f < 0$. Then from the last relation we express y and obtain

$$y = -\frac{e}{f}x - \frac{t}{f}x^3 - 3\frac{u}{f}x^2y - 3\frac{v}{f}xy^2 - \frac{w}{f}y^3. \tag{30}$$

We seek y as a holomorphic function of x. Then we can write

$$y = -\frac{e}{f}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \cdots \tag{31}$$

Substituting (31) into (30) and identifying the coefficients of the same powers of x in the obtained relation we have

$$\begin{gathered}
B_{2n} = 0,\ \forall n \in \mathbb{N},\ B_3 = -\frac{t}{f} + 3\frac{eu}{f^2} - 3\frac{e^2v}{f^3} + \frac{e^3w}{f^4},\\
B_5 = -3\Big(\frac{u}{f} - 2\frac{ev}{f^2} + \frac{e^2w}{f^3}\Big)B_3,\\
B_7 = -3\Big[\Big(\frac{v}{f} - \frac{ew}{f^2}\Big)B_3 - 3\Big(\frac{u}{f} - 2\frac{ev}{f^2} + \frac{e^2w}{f^3}\Big)^2\Big]B_3,\\
B_9 = -\Big[\frac{w}{f}B_3^3 + 6\Big(\frac{v}{f} - \frac{ew}{f^2}\Big)B_3B_5 + 3\Big(\frac{u}{f} - 2\frac{ev}{f^2} + \frac{e^2w}{f^3}\Big)B_7\Big], \ldots
\end{gathered} \tag{32}$$

Substituting (31) into the right-hand side of the critical differential equation (28) we obtain

$$px^3 + 3qx^2y + 3rxy^2 + sy^3 =$$
$$= A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + A_{11}x^{11} + \cdots$$

From this, taking into account (31) and (32) we have

$$\begin{gathered}
A_{2n} = 0,\ \forall n \in \mathbb{N},\ A_3 = p - 3\frac{eq}{f} + 3\frac{e^2r}{f^2} - \frac{e^3s}{f^3},\\
A_5 = 3\Big(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\Big)B_3,\ A_7 = 3\Big[\Big(r - \frac{es}{f}\Big)B_3^2 + \Big(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\Big)B_5\Big],\\
A_9 = sB_3^3 + 6\Big(r - \frac{es}{f}\Big)B_3B_5 + 3\Big(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\Big)B_7,\\
A_{11} = 3\Big[sB_3^2B_5 + 2\Big(r - \frac{es}{f}\Big)B_3B_7 + \Big(r - \frac{es}{f}\Big)B_5^2 + \Big(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\Big)B_9\Big], \ldots
\end{gathered} \tag{33}$$

We introduce the following notations:

$$\begin{gathered}
T = f^3p - 3ef^2q + 3e^2fr - e^3s, \quad U = -f^3t + 3ef^2u - 3e^2fv + e^3w,\\
V = f^2q - 2efr + e^2s, \quad W = fr - es.
\end{gathered} \tag{34}$$

Then, from (32) and (33), we get

$$\begin{gathered}
A_3 = \frac{1}{f^3}T,\ \ B_3 = \frac{1}{f^4}U,\ \ A_5 = \frac{3}{f^2}VB_3,\ \ A_7 = 3(\frac{1}{f}WB_3^2 + \frac{1}{f^2}VB_5),\\
A_9 = sB_3^3 + \frac{6}{f}WB_3B_5 + \frac{3}{f^2}VB_7, \ldots
\end{gathered} \tag{35}$$

Using Theorem 3, the expressions (34) and (35) ($I_1 = f < 0$), we come to the following statement.

Lemma 6. *The stability of unperturbed motion in the system of perturbed motion* (28) *is described by one of the following ten possible cases:*

I. $T < 0$, *then the unperturbed motion is unstable;*

II. $T > 0$, *then the unperturbed motion is stable;*

III. $T = 0$, $UV > 0$, *then the unperturbed motion is unstable;*

IV. $T = 0$, $UV < 0$, *then the unperturbed motion is stable;*

V. $T = V = 0$, $U \neq 0$, $W < 0$, *then the unperturbed motion is unstable;*

VI. $T = V = 0$, $U \neq 0$, $W > 0$, *then the unperturbed motion is stable;*

VII. $T = V = W = 0$, $sU > 0$, *then the unperturbed motion is unstable;*

VIII. $T = V = W = 0$, $sU < 0$, *then the unperturbed motion is stable;*

IX. $T = U = 0$, *then the unperturbed motion is stable;*

X. $p = q = r = s = 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions, moreover, in Cases II, IV, VI and VIII this motion is also asymptotically stable [10]. The expressions T, U, V, W *are given in* (34).

Proof. Assume $A_3 > 0$, then from (35) we get $\frac{T}{f^3} > 0$. Taking into account that $f < 0$, it follows $T < 0$. By Theorem 3 we have proved the Case I. Similarly the Case II is analyzed.

Suppose in (34) that $U \neq 0$. Then from (35) we have $B_3 \neq 0$.

If $A_3 = 0$, i.e. $T = 0$, then by (35) the stability or the instability of unperturbed motion is determined by the sign of expression UV. Then using Theorem 3 we proved the Cases III and IV.

If $T = A_5 = 0$, i.e. $V = 0$, then by (35) the stability or the instability of unperturbed motion is determined according to the sign of expression $\frac{U^2W}{f^9}$. Taking into account that $f < 0$, by Theorem 3 we get the Cases V and VI.

If $A_3 = A_5 = A_7 = 0$ ($T = V = W = 0$), then the stability or the instability of unperturbed motion is determined by the sign of expression A_9, i.e. $\frac{sU}{f^{12}}$. From this, according to Theorem 3, we obtain the Cases VII and VIII. If $T = U = 0$, then all A_k ($k \geq 3$) are equal to zero. By Theorem 3 we have the Case IX. If $U \neq 0$ and $T = V = W = s = 0$, then from (34) we obtain the Case X. Lemma 6 is proved. □

Proceeding from the polynomial bases of center-affine comitants and invariants of the system (27) given in [11], we can write the Sibirsky algebras with generators

$$S_{1,3} = \{J_1, J_2, \dots, J_{20}, K_1, K_2, \dots, K_{13}, Q_1, Q_2, \dots, Q_{14}\}, \quad SI_{1,3} = \{J_1, J_2, \dots, J_{20}\},$$

where J_i, K_j and Q_k are invariants and comitants of these algebras.

For the system (27) we have the notations

$$\begin{aligned} &x^1 = x,\ a^1_1 = c,\ a^1_2 = d,\ a^1_{111} = p,\ a^1_{112} = q,\ a^1_{122} = r,\ a^1_{222} = s,\\ &x^2 = y,\ a^2_1 = e,\ a^2_2 = f,\ a^2_{111} = t,\ a^2_{112} = u,\ a^2_{122} = v,\ a^2_{222} = w. \end{aligned} \quad (36)$$

Further we will need the following generators of Sibirsky algebras $S_{1,3}$ and $SI_{1,3}$, which in tensorial form are written

$$\begin{gathered} J_1 \equiv I_1 = a^\alpha_\alpha, \;\; J_2 \equiv I_2 = a^\alpha_\beta a^\beta_\alpha, \;\; J_3 = a^\alpha_\pi a^\beta_{k\alpha\beta}\varepsilon^{\pi k}, \;\; J_6 = a^\alpha_\pi a^\beta_\gamma a^\gamma_{k\alpha\beta}\varepsilon^{\pi k}, \\ K_1 = a^\alpha_\beta x^\beta x^\gamma \varepsilon_{\alpha\gamma}, \;\; K_2 = a^\alpha_{\alpha\beta\gamma} x^\beta x^\gamma, \;\; K_3 = a^\pi_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma x^k \varepsilon_{\pi k}, \\ Q_1 = a^\pi_\alpha a^k_{\beta\gamma\delta} x^\alpha x^\beta x^\gamma x^\delta \varepsilon_{\pi k}, \;\; Q_2 = a^\alpha_\beta a^\beta_{\alpha\gamma\delta} x^\gamma x^\delta, \\ Q_3 = a^\alpha_\gamma a^\beta_{\alpha\beta\delta} x^\gamma x^\delta, \;\; Q_4 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta\eta} x^\delta x^\eta. \end{gathered} \tag{37}$$

By means of these generators, we compose the following invariant expressions:

$$\begin{gathered} F_1 = K_1(J_6 - J_1 J_3) + J_1[J_1^2 K_2 - J_1(Q_2 + Q_3) + Q_4], \;\; F_2 = J_6 - J_1 J_3, \\ F_3 = K_1[J_3 K_1 - J_1(J_1 K_2 + 2Q_2 - Q_3) + Q_4] + J_1^2(J_1 K_3 + Q_1), \\ F_4 = J_1 K_2 - Q_2, \;\; F_5 = Q_1. \end{gathered} \tag{38}$$

Lemma 7. *Suppose that the first relation from* (7) *is satisfied. Then the system* (27) *by a center-affine transformation can be brought to the form*

$$\frac{dx}{dt} = 0, \;\; \frac{dy}{dt} = ex + fy + tx^3 + 3ux^2y + 3vxy^2 + wy^3$$

if and only if $F_5 \equiv 0$, *where* F_5 *is from* (38).

The proof is similar to Lemma 6. We make use of F_5 which for system (27) has the form

$$\begin{gathered} F_5 = \Delta_{13}(x^1)^4 + (\Delta_{23} + 3\Delta_{14})(x^1)^3 x^2 + 3(\Delta_{15} + \Delta_{24})(x^1)^2(x^2)^2 + \\ +(\Delta_{16} + 3\Delta_{25})x^1(x^2)^3 + \Delta_{26}(x^2)^4, \end{gathered}$$

where Δ_{ij} are the minors of matrix of the coefficients from the right-hand sides of system (27) built on columns i and j of this matrix.

Theorem 5. *Let for differential system of the perturbed motion*

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \;\; (j, \alpha, \beta, \gamma = 1, 2)$$

the invariant conditions $J_1^2 - J_2 = 0$, $J_1 < 0$ *be satisfied. Then the stability of the unperturbed motion is described by one of the following ten possible cases:*

I. $F_1 < 0$, *then the unperturbed motion is unstable;*

II. $F_1 > 0$, *then the unperturbed motion is stable;*

III. $F_1 \equiv 0$, $F_2 F_3 > 0$, *then the unperturbed motion is unstable;*

IV. $F_1 \equiv 0$, $F_2 F_3 < 0$, *then the unperturbed motion is stable;*

V. $F_1 \equiv 0$, $F_2 = 0$, $F_3 \not\equiv 0$, $F_4 < 0$, *then the unperturbed motion is unstable;*

VI. $F_1 \equiv 0$, $F_2 = 0$, $F_3 \not\equiv 0$, $F_4 > 0$, *then the unperturbed motion is stable;*

VII. $F_1 \equiv 0$, $F_2 = 0$, $F_4 \equiv 0$, $F_3 F_5 > 0$, *then the unperturbed motion is unstable;*

VIII. $F_1 \equiv 0$, $F_2 = 0$, $F_4 \equiv 0$, $F_3 F_5 < 0$, *then the unperturbed motion is stable;*

IX. $F_3 \equiv 0$, then the unperturbed motion is stable;

X. $F_5 \equiv 0$, then the unperturbed motion is stable.

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, it is also asymptotically stable in Cases II, IV, VI, VIII. The expressions F_i $(i = \overline{1,5})$ are given in (38).

Proof. Observe that the first three expressions from (38), for critical system (28) with notations (36), look as follows:

$$F_1 = Tx^2, \;\; F_2 = V, \;\; F_3 = Ux^4 + Tx^3y. \tag{39}$$

Suppose that $F_1 \equiv 0,\; F_2 = 0$. Then by means of the polynomials T, V, W from (34), we get for expression F_4 from (38) that $F_4 = W(\frac{e}{f}x + y)^2$. Using the expressions (39), the last assertion together with Lemmas 6 and 7, we obtain the Cases I-X. We note that the comitants F_1, F_2F_3, F_4, F_3F_5 from (38), used in the Cases I-VIII of Theorem 5, are even-degree comitants with respect to x and y and have the weights [2] equal to $0, 0, 0, -2$, respectively. Moreover, each one of these comitants (in the case when it is applied) is a binary form with a well defined sing. This ensures that any center-affine transformation cannot change their sign. Theorem 5 is proved. □

5 Critical system of Lyapunov type with nonlinearities of degree four

We consider the differential system of perturbed motion with polynomial nonlinearities

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4, \\ \frac{dy}{dt} &= ex + fy + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4, \end{aligned} \tag{40}$$

where $c, d, e, f, g, h, k, l, m, n, p, q, r, s$ are real arbitrary coefficients.

Similar to the previous cases, when the characteristic equation of (40) has one zero root and the other one is negative, i.e. the conditions (7) are satisfied, then this system by a center-affine transformation can be brought to its critical form

$$\begin{aligned} \frac{dx}{dt} &= gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4, \\ \frac{dy}{dt} &= ex + fy + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4. \end{aligned} \tag{41}$$

According to Theorem 3, we analyze the equation

$$ex + fy + nx^4 + 4px^3y + 6qx^2y^2 + 4rxy^3 + sy^4 = 0. \tag{42}$$

As for system (40) we have $I_1 = f < 0$, then from (42) we express y:

$$y = -\frac{e}{f}x - \frac{n}{f}x^4 - 4\frac{p}{f}x^3y - 6\frac{q}{f}x^2y^2 - 4\frac{r}{f}xy^3 - \frac{s}{f}y^4. \tag{43}$$

We seek y as a holomorphic function of x. Then we can write

$$\begin{aligned} y = -\frac{e}{f}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \\ + B_{10}x^{10} + B_{11}x^{11} + B_{12}x^{12} + B_{13}x^{13} + B_{14}x^{14} + B_{15}x^{15} + B_{16}x^{16} + \cdots \end{aligned} \tag{44}$$

Substituting (44) into (43) and equating the coefficients of monomials in x, we find that

$$\begin{aligned} &B_i = 0 \ (i = 2,3,5,6,8,9,11,12,14,15,\dots), \quad B_4 = -\Big(\frac{n}{f} - 4\frac{ep}{f^2} + 6\frac{e^2q}{f^3} - 4\frac{e^3r}{f^4} + \frac{e^4s}{f^5}\Big), \\ &B_7 = -4\Big(\frac{p}{f} - 3\frac{eq}{f^2} + 3\frac{e^2r}{f^3} - \frac{e^3s}{f^4}\Big)B_4, \\ &B_{10} = -2\Big[3\Big(\frac{q}{f} - 2\frac{er}{f^2} + \frac{e^2s}{f^3}\Big)B_4^2 + 2\Big(\frac{p}{f} - 3\frac{eq}{f^2} + 3\frac{e^2r}{f^3} - \frac{e^3s}{f^4}\Big)B_7\Big], \\ &B_{13} = -4\Big[\Big(\frac{r}{f} - \frac{es}{f^2}\Big)B_4^3 + 3\Big(\frac{q}{f} - 2\frac{er}{f^2} + \frac{e^2s}{f^3}\Big)B_4B_7 + \\ &\quad + \Big(\frac{p}{f} - 3\frac{eq}{f^2} + 3\frac{e^2r}{f^3} - \frac{e^3s}{f^4}\Big)B_{10}\Big], \\ &B_{16} = -\Big[\frac{s}{f}B_4^4 + 12\Big(\frac{r}{f} - \frac{es}{f^2}\Big)B_4^2B_7 + 6\Big(\frac{q}{f} - 2\frac{er}{f^2} + \frac{e^2s}{f^3}\Big)\Big(2B_4B_{10} + B_7^2\Big) + \\ &\quad + 4\Big(\frac{p}{f} - 3\frac{eq}{f^2} + 3\frac{e^2r}{f^3} - \frac{e^3s}{f^4}\Big)B_{13}\Big], \dots \end{aligned} \tag{45}$$

Substituting (44) into the right-hand side of the critical differential equation, then from (41) we get

$$gx^4 + 4hx^3y + 6kx^2y^2 + 4lxy^3 + my^4 = A_2x^2 + A_3x^3 + \cdots + A_{16}x^{16} + \cdots$$

Hence, taking into account (44) and (45) we have

$$\begin{aligned} &A_i = 0 \ (i = 2,3,5,6,8,9,,11,12,14,15,\dots), \quad A_4 = g - 4\frac{eh}{f} + 6\frac{e^2k}{f^2} - 4\frac{e^3l}{f^3} + \frac{e^4m}{f^4}, \\ &A_7 = 4\Big(h - 3\frac{ek}{f} + 3\frac{e^2l}{f^2} - \frac{e^3m}{f^3}\Big)B_4, \\ &A_{10} = 2\Big[3\Big(k - 2\frac{el}{f} + \frac{e^2m}{f^2}\Big)B_4^2 + 2\Big(h - 3\frac{ek}{f} + 3\frac{e^2l}{f^2} - \frac{e^3m}{f^3}\Big)B_7\Big], \\ &A_{13} = 4\Big[\Big(l - \frac{em}{f}\Big)B_4^3 + 3\Big(k - 2\frac{el}{f} + \frac{e^2m}{f^2}\Big)B_4B_7 + \\ &\quad + \Big(h - 3\frac{ek}{f} + 3\frac{e^2l}{f^2} - \frac{e^3m}{f^3}\Big)B_{10}\Big], \\ &A_{16} = mB_4^4 + 12\Big(l - \frac{em}{f}\Big)B_4^2B_7 + 6\Big(k - 2\frac{el}{f} + \frac{e^2m}{f^2}\Big)\Big(2B_4B_{10} + B_7^2\Big) + \\ &\quad + 4\Big(h - 3\frac{ek}{f} + 3\frac{e^2l}{f^2} - \frac{e^3m}{f^3}\Big)B_{13}, \dots \end{aligned} \tag{46}$$

Let us introduce the following notation:

$$\begin{gathered} A = f^4g - 4ef^3h + 6e^2f^2k - 4e^3fl + e^4m, \\ B = -f^4n + 4ef^3p - 6e^2f^2q + 4e^3fr - e^4s, \\ C = f^3h - 3ef^2k + 3e^2fl - e^3m, \quad D = f^2k - 2efl + e^2m, \quad E = fl - em. \end{gathered} \tag{47}$$

Then taking into account (47), we obtain from (45)–(46) that

$$\begin{gathered} A_4 = \frac{1}{f^4}A, \quad B_4 = \frac{1}{f^5}B, \quad A_7 = \frac{4}{f^8}BC, \quad A_{10} = 2\Big(\frac{3}{f^{12}}B^2D + \frac{2}{f^3}CB_7\Big), \\ A_{13} = 4\Big(\frac{1}{f^{16}}B^3E + \frac{3}{f^2}DB_4B_7 + \frac{1}{f^3}CB_{10}\Big), \\ A_{16} = mB_4^4 + \frac{12}{f}EB_4^2B_7 + \frac{6}{f^2}D(2B_4B_{10} + B_7^2) + \frac{4}{f^3}CB_{13}, \dots \end{gathered} \tag{48}$$

Lemma 8. *The stability of unperturbed motion in the system of perturbed motion* (41) *is described by nine possible cases, if for expressions* (47) *(*$I_1 = f < 0$*) the following conditions are satisfied:*

I. $A \neq 0$, *then the unperturbed motion is unstable;*

II. $A = 0, \;\; BC > 0$, *then the unperturbed motion is unstable;*

III. $A = 0, \;\; BC < 0$, *then the unperturbed motion is stable;*

IV. $A = C = 0, \;\; BD \neq 0$, *then the unperturbed motion is unstable;*

V. $A = C = D = 0, \;\; BE > 0$, *then the unperturbed motion is unstable;*

VI. $A = C = D = 0, \;\; BE < 0$, *then the unperturbed motion is stable;*

VII. $A = C = D = E = 0, \;\; mB \neq 0$, *then the unperturbed motion is unstable;*

VIII. $A = B = 0$, *then the unperturbed motion is stable;*

IX. $g = h = k = l = m = 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, this motion is also asymptotically stable [10] in Cases III and VI. The expressions A, B, C, D, E *are given in* (47)*.*

Proof. If $A_4 \neq 0$, then from (48) we have $A \neq 0$. By Theorem 3, we get the Case I.

Suppose in (47) that $B \neq 0$. Then (48) implies that $B_4 \neq 0$. If $A_4 = 0$, i.e. $A = 0$, then according to (48) the stability or the instability of unperturbed motion is determined by the sign of the expression A_7 (the sign of the product BC). Using Theorem 3 we obtain the Cases II and III.

When $A = A_7 = 0$, i.e. $C = 0$, then from (48) we have $A_{10} = \frac{6}{f^{12}}B^2D$. If $D \neq 0$, then we obtain the Case IV (see Theorem 3).

Suppose $A = C = D = 0$. Then from (48) it results that $A_{13} \neq 0$, when $BE \neq 0$. So the stability or the instability of the unperturbed motion is determined by the sign of expression BE. Using Theorem 3 we get the Cases V and VI.

When $A_4 = A_7 = A_{10} = A_{13} = 0$ $(B \neq 0)$, then we have $A = C = D = E = 0$. If $A_{16} \neq 0$, then from (48) we obtain the Case VII. If $A = B = 0$, then all A_k $(k \geq 4)$ vanish. By Theorem 3 we get the Case VIII. If $A = C = D = E = 0$ and $m = 0$, then (47) with $f < 0$ implies the Case IX. Lemma 8 is proved. □

Let φ and ψ be homogeneous comitants of degree ρ_1 and ρ_2 respectively of the phase variables x and y of a two-dimensional polynomial differential system. Then by [3] the transvectant

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1-j)(\rho_2-j)}{\rho_1!\rho_2!}\sum_{i=0}^{j}(-1)^j\begin{pmatrix} j \\ i \end{pmatrix}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}} \tag{49}$$

is also a comitant for this system.

In the Iu. Calin's works, see for example [12], it is shown that by means of the transvectant (49) all generators of the Sibirsky algebras of comitants and invariants for any system of type (1) can be constructed.

We denote the homogeneities from the right-hand sides of system (40) as follows:

$$\begin{aligned} &P_1(x,y) = cx+dy, \;\; P_4(x,y) = gx^4+4hx^3y+6kx^2y^2+4lxy^3+my^4,\\ &Q_1(x,y) = ex+fy, \;\; Q_4(x,y) = nx^4+4px^3y+6qx^2y^2+4rxy^3+sy^4. \end{aligned} \tag{50}$$

According to [13], we write the following comitants of the system (40)

$$R_i = P_i(x,y)y - Q_i(x,y)x, \quad S_i = \frac{1}{i}\Big(\frac{\partial P_i(x,y)}{\partial x}+\frac{\partial Q_i(x,y)}{\partial y}\Big), \;\; (i=1,4). \tag{51}$$

Later on, we will need the following comitants and invariants from [13] of system (40) built by operations (49) and (51):

$$\begin{gathered} I_1 = S_1, \;\; I_2 = (R_1,R_1)^{(2)}, \;\; K_1 = R_4, \;\; K_2 = S_4, \;\; Q_1 = R_1, \;\; Q_2 = S_1,\\ Q_3 = (R_4,R_1)^{(2)}, \;\; Q_4 = (R_4,R_1)^{(1)}, \;\; Q_5 = (S_4,S_1)^{(2)}, \;\; Q_6 = (S_4,R_1)^{(1)},\\ Q_{19} = [\![R_4,R_1)^{(2)},R_1)^{(2)}, \;\; Q_{20} = [\![R_4,R_1)^{(2)},R_1)^{(1)},\\ Q_{21} = [\![S_4,R_1)^{(2)},R_1)^{(1)}, \;\; Q_{43} = [\![R_4,R_1)^{(2)},R_1)^{(2)},R_1)^{(1)}, \end{gathered} \tag{52}$$

where the sign “$[\![$” denotes all the parentheses of the transvectant that have to be written in the left.

We consider for system (40) the following expressions composed of comitants and invariants from (52) that can be written in the form:

$$\begin{gathered} H_1 = Q_1[Q_2(15Q_{19}-8Q_{21})-10Q_{43}+12I_1^2Q_5]+Q_2^2[Q_2(4K_2Q_2+5Q_3-8Q_6)-10Q_{20}],\\ H_2 = 5Q_2^3(K_1Q_2-2Q_4)+2Q_1^2(5Q_{19}+4Q_{21}-6Q_2Q_5)-4Q_1Q_2[Q_2(K_2Q_2-5Q_3-2Q_6)+\\ +5Q_{20}], \;\; H_3 = Q_2(5Q_{19}-6Q_{21}+3Q_2Q_5)-10Q_{43}, \;\; H_4 = 5I_1Q_5+10Q_{19}-2Q_{21},\\ H_5 = Q_1, \;\; H_6 = 5I_1K_2+10Q_3-6Q_6, \;\; H_7 = 8K_2Q_1-5K_1Q_2-10Q_4. \end{gathered} \tag{53}$$

Lemma 9. *Suppose that the first equality holds in* (7)*. Then by a center-affine transformation the system* (40) *can be brought to the form*

$$\frac{dx}{dt} = 0, \;\; \frac{dy}{dt} = ex+fy+nx^4+4px^3y+6qx^2y^2+4rxy^3+sy^4$$

if and only if the condition $H_7 \equiv 0$ is satisfied, where H_7 is from (53)*.*

The proof of this Lemma is similar to Lemma 6. Here, the fact is used that H_7 from (53), for the system (40), is of the form

$$H_7 = 10[\Delta_{13}(x^1)^5 + (\Delta_{23} + 4\Delta_{14})(x^1)^4x^2 + 2(2\Delta_{24} + 3\Delta_{15})(x^1)^3(x^2)^2+$$
$$+2(2\Delta_{16} + 3\Delta_{25})(x^1)^2(x^2)^3 + (\Delta_{17} + 4\Delta_{26})x^1(x^2)^4 + \Delta_{27}(x^2)^5],$$

where Δ_{ij} are the minors of the matrix of coefficients from the right-hand sides of system (40), built on the columns i and j of this matrix.

Theorem 6. *Let for system of perturbed motion* (40) *the invariant conditions* (7) *be satisfied. Then the stability of the unperturbed motion is described by one of the following nine possible cases:*

I. $H_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $H_1 \equiv 0$, $H_2H_3 > 0$, *then the unperturbed motion is unstable;*

III. $H_1 \equiv 0$, $H_2H_3 < 0$, *then the unperturbed motion is stable;*

IV. $H_1 \equiv H_3 \equiv 0$, $H_2H_4 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $H_1 \equiv H_3 \equiv H_4 \equiv 0$, $H_2H_5H_6 > 0$, *then the unperturbed motion is unstable;*

VI. $H_1 \equiv H_3 \equiv H_4 \equiv 0$, $H_2H_5H_6 < 0$, *then the unperturbed motion is stable;*

VII. $H_1 \equiv H_3 \equiv H_4 \equiv H_6 \equiv 0$, $H_2H_7 \not\equiv 0$, *then the unperturbed motion is unstable;*

VIII. $H_2 \equiv 0$, *then the unperturbed motion is stable;*

IX. $H_7 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motions, and moreover in Cases III, and VI this motion is also asymptotically stable [2]. The expressions H_i $(i = \overline{1,7})$ *are given in* (53).

Proof. The first three expressions from (53), for the critical system (41), give

$$H_1 = 10Ax^3,\ H_2 = 10Bx^5 + 10Ax^4y,\ H_3 = 10Cx. \tag{54}$$

Next the proof is based on Lemma 8. The Case I is obvious if we use (54). Put $H_1 = 0$, then by Lemma 8, from (54) we obtain the Cases II and III.

The product H_2H_3 is of even degree with respect to x and has the weight equal to 0 [2]. Therefore, the expression H_2H_3 under any center-affine transformation does not change its sign. Using (54), the Case IV of Lemma 8 implies the Case IV of Theorem 6 and we have $(f = I_1 < 0)$

$$H_2 = 10Bx^5,\ H_4 = 10D(\frac{e}{f}x + y). \tag{55}$$

For the Cases V and VI of the Theorem, Lemma 8 yields $A = C = D = 0$. Then from (54) and (55), we obtain the invariant equations for the examined cases. By means of these equations and the expressions from (53), we obtain

$$H_2 = 10Bx^5,\ \ H_5 = -fx\Big(\frac{e}{f}x + y\Big),\ \ H_6 = 10E\Big(\frac{e}{f} + y\Big)^3. \tag{56}$$

From this, we get $H_2H_5H_6 = -100fBEx^6(\frac{e}{f}x+y)^4$. This product is of even degree with respect to x and y and have the weight -2 and has a well defined sign. Hence, we have the Cases V-VI.

The Case VII of Theorem 6 is obtained by using the Case VII of Lemma 8 and the expressions (54)–(56). Indeed, for this case we have

$$H_7 = -10fm\Big(\frac{e}{f}x+y\Big)^4.$$

By means of H_7 and H_2 from (54), we get the Case VII with inequality H_2H_7.

The Case VIII of the Theorem results from (54) using the Case VIII of Lemma 8 and expressions (47)–(48). The Case IX results from the Case IX of Lemma 8 and the assertion of Lemma 9. Theorem 6 is proved. □

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