SOME PROPERTIES OF LEFT PRODUCT OF TWO SUBCATEGORIES

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Abstract. We study some properties of left product of two subcategories: one coreflective and one reflective in the category of local convex topological vectorial Hausdorff spaces. In this work on examined the situation generated by a structures of factorization $(\mathcal{P}'', \mathcal{I}'')$ with certain properties, allowing to prove that the left product of some coreflective subcategories with any $\mathcal{P}''$ - reflective subcategory is one and the same. In addition, be indicated examples of coreflectors and reflectors functors which commutes.
Key words: coreflective and reflective subcategory, left product of two subcategories, coreflective subcategory of the topological Mackey spaces, subcategory of spaces with weak topology.

UNELE PROPRIETĂŢI ALE PRODUSULUI DE STÂNGA A DOUĂ SUBCATEGORII

Rezumat. Vom studia unele proprietăţi ale produsului de stânga a două subcategorii: una coreflectivă şi una reflectivă din categoria spaţiilor topologice Hausdorff vectoriale local convexe. În acest articol se va examina situaţia generată de structurile de factorizare $(\mathcal{P}'', \mathcal{I}'')$ cu anumite proprietăţi, care va permite să demonstrăm că produsul de stânga a unor subcategorii coreflective cu orice $\mathcal{P}''$ - subcategorie reflectivă este una şi aceiaşi. În plus, vor fi indicate example de functori coreflectori şi reflectori care comută.
Cuvinte cheie: subcategorie coreflectivă şi reflectivă , produsul de stânga a două subcategorii, subcategoria coreflectivă a spaţiilor topologice Mackey, subcategoria spaţiilor cu topologie slabă.

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In category $\mathcal{C}_2\mathcal{V}$ of the local convex topological vectorial Hausdorff spaces are studied the properties of the left product of two subcategories $\mathcal{K} *_s \mathcal{R}$ - one coreflective $\mathcal{K}$ and one reflective $\mathcal{R}$. On indicate sufficient conditions, that this product should be a coreflective subcategory (Theorem 2). We indicate examples when this product is not a coreflective subcategory (Proposition 1). We denote:

$\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi | r(e) \in \mathcal{I}so\}$, and $\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid k(m) \in \mathcal{I}so\}$,

where $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ and $k : \mathcal{C}_2\mathcal{V} \to \mathcal{K}$ are the respective functors. It is known that the $((\varepsilon\mathcal{R}) \circ \mathcal{E}_p, ((\varepsilon\mathcal{R}) \circ \mathcal{E}_p^{\downarrow}))$ which we will note $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$ or $(\mathcal{P}'', \mathcal{I}'')$ is a structure of factorization in $\mathcal{C}_2\mathcal{V}$, (to see [1]). Here $(\mathcal{E}_p, \mathcal{M}_u)$ is a structure of factorization defined by class $\mathcal{M}_u$ of universal monomorphisms (to see [1], [4]).

$\mathcal{R}$ is the smallest element in the class of $\mathcal{P}''$-reflective subcategories and there is the smallest element $\mathcal{M}$ in the $\mathbb{K}(\mathcal{I}'')$ class of $\mathcal{I}''$-reflective subcategories. For any $\mathcal{K} \in \mathbb{K}(\mathcal{I}'')$ and anything $\mathcal{R}_1 \in \mathbb{R}(\mathcal{P}'')$ we have $\mathcal{K} = \mathcal{K} *_{sr} \mathcal{R}_1$ (Theorem 3). It is demonstrates that $\overline{\mathcal{M}} = \tilde{\mathcal{M}} *_s \mathcal{R}$, where $\tilde{\mathcal{M}}$ is the subcategory of Mackey spaces(Theorem 7). If $\mathcal{R}$ contains subcategory $\mathcal{S}$ of the spaces with weak topology, then functors $\overline{m} : \mathcal{C}_2\mathcal{V} \longrightarrow \overline{\mathcal{M}}$ and $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $\overline{m} \cdot r = r \cdot \overline{m}$ (Corollary 2).

We will use the following notation.

The structure of factorization:

$(\mathcal{E}_u, \mathcal{M}_p)$ = (the class of universal epimorphisms, the class of precise monomorphisms) = (the class of surjective applications, the class of topological inclusions);

$(\mathcal{E}_p, \mathcal{M}_u)$ = (the class of precise epimorphisms, the class of universal monomorphisms)(to see [1]);

The coreflective and reflective subcategory:

$\tilde{\mathcal{M}}$ - the coreflective subcategory of spaces with Mackey topology;

Σ - the coreflective subcategory of the spaces with the strongest locally convex topology;

$\mathcal{S}$ - the subcategory of spaces with weak locally convex topology;

Π - the subcategory of complete spaces with a weak topology;

$\mathbb{K}$ - the class of nonzero coreflective subcategories;

$\mathbb{R}$ - the class of nonzero reflective subcategories.

Concerning the notions and notations in category $\mathcal{C}_2\mathcal{V}$ see [6].

Either $\mathcal{B}$ a class of bimorphisms. We denote $\mathbb{K}(\mathcal{B})$, (respectively $\mathbb{R}(\mathcal{B})$) - the class of $\mathcal{B}$-coreflective subcategories (respectively $\mathcal{B}$-reflective).

Let $\mathcal{K}$ be is a epicoreflective subcategory, and $\mathcal{R}$ is a monoreflective subcategory of category $\mathcal{C}$ with corresponding functors $k : \mathcal{C} \to \mathcal{K}$ and $r : \mathcal{C} \to \mathcal{R}$. For any object X of category $\mathcal{C}$ either $k^X : kX \to X$ and $r^X : X \to rX$ $\mathcal{K}$-coreplica and $\mathcal{R}$-replica to this object. Further, either $r^{kX} : kX \to rkX$ $\mathcal{R}$-replica of object kX, and $r(k^X) : rkX \to rX$ that unique morphism for which

$$r(k^X) \cdot r^{kX} = r^X \cdot k^X. \qquad (1)$$

On morphisms r^X and $r(k^X)$ we build the pull-back square

$$r^X \cdot w^X = r(k^X) \cdot f^X. \qquad (2)$$

From equality (1) there exists a morphism t^X so that

$$w^X \cdot t^X = k^X, \qquad (3)$$

$$f^X \cdot t^X = r^{kX}. \qquad (4)$$

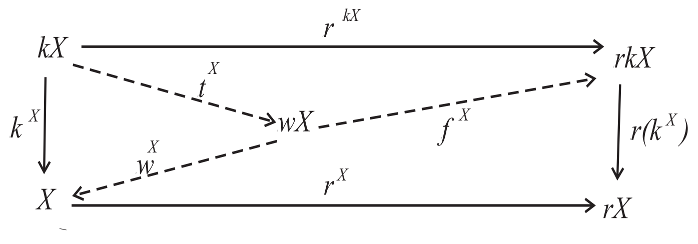


Diagram of the left product (PS)

We denote by $\mathcal{W} = \mathcal{K} *_s \mathcal{R}$ the full subcategory of category $\mathcal{C}$ consisting of all objects isomorphic to objects form wX.

Definition 1. The subcategory $\mathcal{W} = \mathcal{K} *_s \mathcal{R}$ is called the s-product or the left product of subcategories $\mathcal{K}$ and $\mathcal{R}$.

Duality is defined the right product of two subcategories $\mathcal{V} = \mathcal{K} *_d \mathcal{R}$.

Check easy as correspondence $X \mapsto wX$ defines a functor $w : \mathcal{C} \to \mathcal{W}$. We shall say that $\mathcal{W}$ is a coreflective subcategory, if w is an coreflector functor.

Examples have been constructed, showing that the left product of two subcategories is not a coreflective subcategory. Thereby emerged necessity to find sufficient conditions when the left product is a coreflective subcategory. In the paper [3] were established a series of necessary and sufficient conditions for that left product to be a coreflective subcategory.

The following theorems indicates sufficient conditions for that left product to be a coreflective subcategory, and the right product of two subcategories to be a reflective subcategory.

THEOREM 1. *1. Let $\mathcal{K}$ be a $\mathcal{M}_u$-coreflective subcategory of the category $\mathcal{C}$. Then for any reflective subcategory $\mathcal{R}$ of the category $\mathcal{C}$ we have:*

a) *the left product $\mathcal{K} *_s \mathcal{R}$ is a $\mathcal{M}_u$-coreflective subcategory of category $\mathcal{C}$;*

b) *the right product $\mathcal{K} *_d \mathcal{R}$ is a reflective subcategory of category $\mathcal{C}$.*

1^0. Let $\mathcal{R}$ be a $\mathcal{E}_u$-reflective subcategory of category $\mathcal{C}$. Then for any coreflective subcategory $\mathcal{K}$ of category $\mathcal{C}$ we have:

a) *the right product $\mathcal{K} *_d \mathcal{R}$ is a $\mathcal{E}_u$-reflective subcategory of category $\mathcal{C}$;*

b) *the left product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory of category $\mathcal{C}$.*

For the category $\mathcal{C}_2\mathcal{V}$ previous theorem can be formulated as such:

THEOREM 2. *1. Let $\mathcal{K}$ be is a coreflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and $\tilde{\mathcal{M}} \subset \mathcal{K}$. Then for any reflective subcategory $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$ we have:*

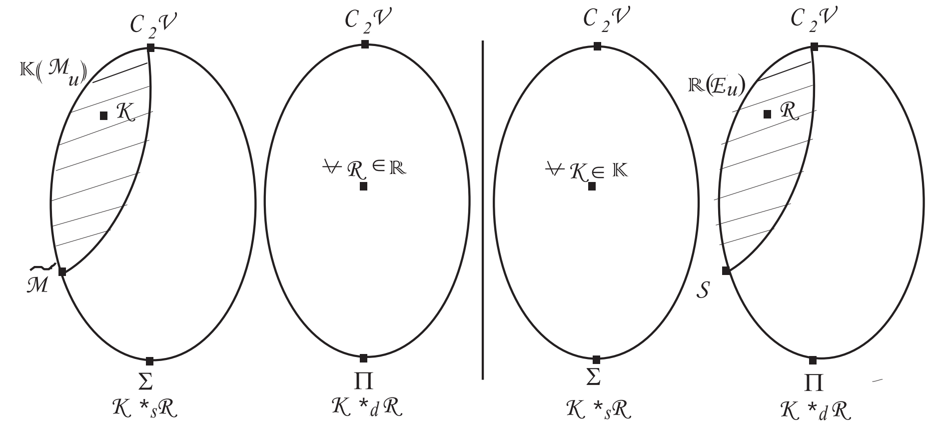
a) the left product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and $\tilde{\mathcal{M}} \subset \mathcal{K} *_s \mathcal{R}$;

b) the right product $\mathcal{K} *_d \mathcal{R}$ is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$.

1^0. Let $\mathcal{R}$ be is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and $\mathcal{S} \subset \mathcal{R}$. Then for any coreflective subcategory $\mathcal{K}$ of category $\mathcal{C}_2\mathcal{V}$ we have:

a) the right product $\mathcal{K} *_d \mathcal{R}$ is a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and $\mathcal{S} \subset \mathcal{K} *_d \mathcal{R}$;

b) the left product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory of category $\mathcal{C}_2\mathcal{V}$.

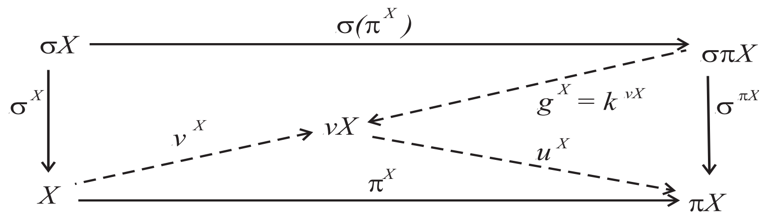


The above diagram indicates the cases when the left product is a coreflective subcategory, and cases where the right product is a reflective subcategory. The left product is not always a reflective subcategory. In category $\mathcal{C}_2\mathcal{V}$ and in the category Th of Tihonov spaces there are examples when this product is not a reflective subcategory (see [5]).

PROPOSITION 1. *In the category* $\mathcal{C}_2\mathcal{V}$

1. *The right product* $\Sigma *_d \Pi$ *is not a reflective subcategory.*
2. *The left product* $\Sigma *_s \Pi$ *is not a coreflective subcategory.*

Proof. 1. We build the following diagram for object X does not belong to subcategory Π.



Because π^X is un *mono*, it results as well $\sigma(\pi^X)$ it's the same. Hence $\sigma(\pi^X)$ is sectionalized. So and v^X is sectionalized. If v^X is un *epi*, then it is un *iso*. In this case $\pi^X \in \mathcal{E}_u \cap \mathcal{M}_u$, i.e. π^X is un *iso*.

2. Is demonstrated in an analogous manner. □

In cases where the left or right product is not a coreflective (respectively reflective) subcategory recourse is had to the factorization of these products (see [5]).

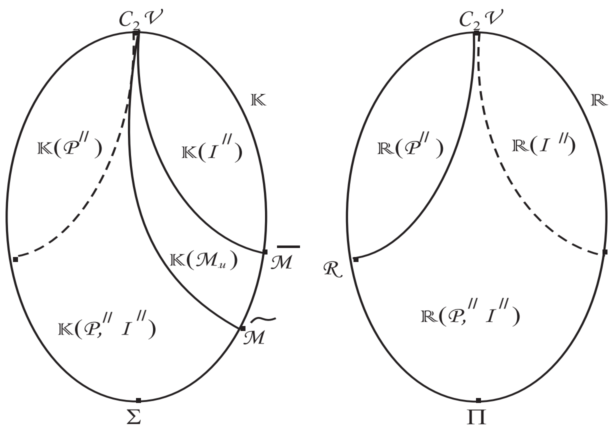
Let $\mathcal{R}$ be a nonzero reflective subcategory of category $\mathcal{C}_2\mathcal{V}$, for which we fix the structure of factorization $(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$, where $\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \circ \mathcal{E}_p$ (see [1]). This structure divides both the lattice $\mathbb{K}$ of nonzero coreflective subcategories, as well as lattice $\mathbb{R}$ of nonzero reflective subcategories in three classes:

$\mathbb{K}$ - $\mathbb{K}(\mathcal{P}'')$, $\mathbb{K}(\mathcal{I}'')$, $\mathbb{K}(\mathcal{P}'', \mathcal{I}'')$;

$\mathbb{R}$ - $\mathbb{R}(\mathcal{P}'')$, $\mathbb{R}(\mathcal{I}'')$, $\mathbb{R}(\mathcal{P}'', \mathcal{I}'')$.

where $\mathbb{K}(\mathcal{P}'') = \{\mathcal{K} \in \mathbb{K} | \mathcal{K}$ is $\mathcal{P}''$-coreflective$\}$, $\mathbb{K}(\mathcal{I}'') = \{\mathcal{K} \in \mathbb{K} | \mathcal{K}$ is $\mathcal{I}''$-coreflective$\}$, $\mathbb{K}(\mathcal{P}'', \mathcal{I}'') = \{\mathbb{K} \setminus (\mathbb{K}(\mathcal{P}'') \cup \mathbb{K}(\mathcal{I}''))\} \cup \{\mathcal{C}_2\mathcal{V}\}$, and analogue division of lattice $\mathbb{R}$.

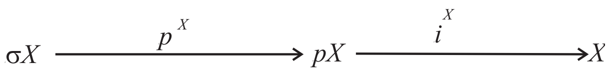
LEMMA 1. $\mathbb{K}(\mathcal{I}'') \subset \mathbb{K}(\mathcal{M}_u)$.



Proof. Because $\mathcal{I}'' \subset \mathcal{M}_u$, it follows that any $\mathcal{I}''$-coreflective subcategory it is also $\mathcal{M}_u$-coreflective. It remains to be remembered that $\tilde{\mathcal{M}}$ is the smallest element in the class $\mathbb{K}(\mathcal{M}_u)$. □

Thus class $\mathbb{K}(\mathcal{I}'')$ possess the smallest element that we will write $\overline{\mathcal{M}}$ and the class $\mathbb{R}(\mathcal{P}'')$ - the smallest element $\mathcal{R}$. The $\overline{\mathcal{M}}$-coreplique of an object X is obtained by performing $(\mathcal{P}'', \mathcal{I}'')$-factorization of the Σ-coreplique $\sigma^X : \sigma X \to X$

$$\sigma^X = i^X \cdot p^X.$$

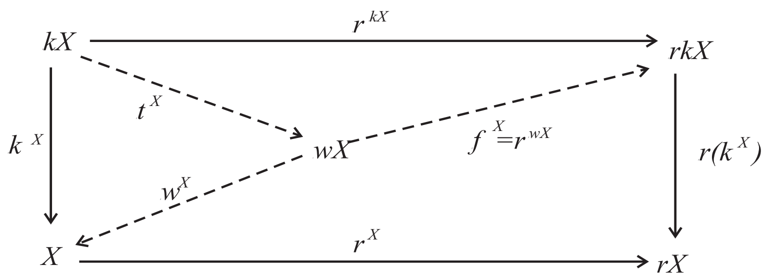


Then i^X is $\overline{\mathcal{M}}$-coreplica of the object X.

LEMMA 2. *Let $\mathcal{K} \in \mathbb{K}$ be a coreflective subcategory for which the product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory, and $(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$. The following statements are equivalent:*

1. $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$.
2. $\mathcal{K} *_s \mathcal{R} \in \mathbb{K}(\mathcal{I}'')$.

Proof. $1 \Rightarrow 2$. Because $\tilde{\mathcal{M}} \subset \mathcal{K}$, the product $\mathcal{K} *_s \mathcal{R}$ is a coreflective subcategory (Theorem 2). Let's build the diagram (PS) for un arbitrary object X of category $\mathcal{C}_2\mathcal{V}$.



In equality

$$k^X = w^X \cdot t^X \tag{1}$$

$k^X \in \mathcal{M}_u$, and $t^X \in \mathcal{E}pi$. Because class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary (see [1]), we deduce that $w^X \in \mathcal{M}_u$. So

$$r^X \cdot w^X = r(k^X) \cdot r^{wX}, \tag{2}$$

is an pull-back square and $w^X \in \mathcal{M}_u$. According to the Theorem 7.3 [4] $w^X \in \mathcal{I}'' = \mathcal{I}''(\mathcal{R})$.

$2 \Rightarrow 1$. In equality

$$r^{kX} = r^{wX} \cdot t^X \tag{3}$$

$r^{kX} \in \mathcal{M}_u$. So and $t^X \in \mathcal{M}_u$. Thus $w^X \in \mathcal{I}'' \subset \mathcal{M}_u$ and $t^X \in \mathcal{M}_u$. From equality (1) it results that $k^X \in \mathcal{M}_u$. □

THEOREM 3. *Let $\mathcal{K}$ be a coreflective subcategory of category $\mathcal{C}_2\mathcal{V}$, $\tilde{\mathcal{M}} \subset \mathcal{K}$, and $(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$. Then the following statements are equivalent:*

1. $\mathcal{K} \in \mathbb{K}(\mathcal{I}'')$.
2. $\mathcal{K} = \mathcal{K} *_s \mathcal{R}$.
3. *For any element $\mathcal{R}_1 \in \mathbb{R}(\mathcal{P}'')$, we have $\mathcal{K} = \mathcal{K} *_s \mathcal{R}_1$.*

Proof. We will demonstrate the following implications: $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$.

$1 \Longrightarrow 2$. For arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ let $k^X : kX \to X$ $\mathcal{K}$-coreplica be, and $r^X : X \to rX$ and $r^{kX} : kX \to rkX$ the $\mathcal{R}$-replicas of the respectively objects. We have the commutative square

$$r^X \cdot k^X = r(k^X) \cdot r^{kX}. \tag{1}$$

Because $\mathcal{K} \in \mathbb{K}(\mathcal{I}'')$, it results that $k^X \in \mathcal{I}''$, and the square (1) is pull-back (Theorem 7.3 [4]). So $\mathcal{K} = \mathcal{K} *_s \mathcal{R}$.

2 $\Longrightarrow$ 3. Let $\mathcal{K} = \mathcal{K} *_s \mathcal{R}$ and $\mathcal{R}_1 \in \mathbb{R}(\mathcal{P}'')$ be. For the object $X \in |\mathcal{C}_2\mathcal{V}|$ let be $k^X : kX \to X$ $\mathcal{K}$-coreplica, $r_1^X : X \to r_1X$ and $r_1^{kX} : kX \to r_1kX$ $\mathcal{R}_1$-replicas of respective objects, and $r^X : X \to rX$ and $r^{kX} : kX \to rkX$ $\mathcal{R}$-replicas of respective objects. Because $\mathcal{R} \subset \mathcal{R}_1$ we deduce that

$$r^{kX} = f^{r_1kX} \cdot r_1^{kX}, \tag{2}$$

$$r^X = f^{r_1X} \cdot r_1^X, \tag{3}$$

for two morphisms f^{r_1kX} şi f^{r_1X}. We still have the equals:

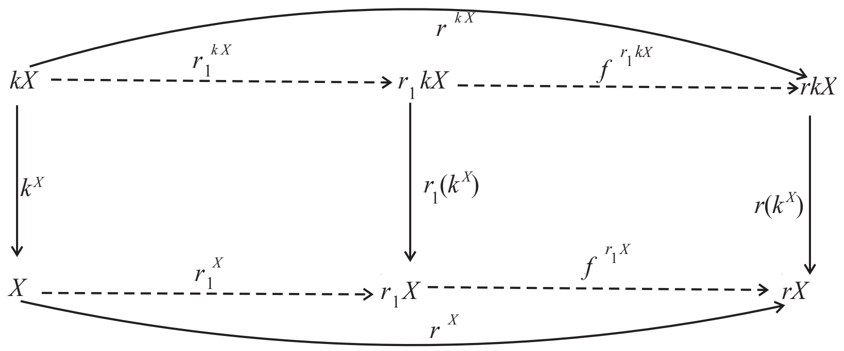
$$r_1^X \cdot k^X = r_1(k^X) \cdot r_1^{kX}, \tag{4}$$

$$r^X \cdot k^X = r(k^X) \cdot r^{kX}. \tag{5}$$

From these results we have and the equality

$$f^{r_1X} \cdot r_1(k^X) = r(k^X) \cdot f^{r_1kX}. \tag{6}$$

According to the hypothesis, the square (5) is pull-back, and in this diagram all morphisms are bimorphisms. It is easy to check that square (4) is also pull-back.



3 $\Longrightarrow$ 1. Because $\mathcal{K} = \mathcal{K} *_s \mathcal{R}$, it results that the square

$$r^X \cdot k^X = r(k^X) \cdot r^{kX} \tag{7}$$

is pull-back. On the other hand $\tilde{\mathcal{M}} \subset \mathcal{K}$. So $k^X \in \mathcal{M}_u$. Therefore $k^X \in \mathcal{I}''$, and $\mathcal{K} \in \mathbb{K}(\mathcal{I}'')$. □

The reflective subcategory $\mathcal{R}$ establishes the following relationship of $\mathcal{R}$-equivalence in the class $\mathbb{K}(\mathcal{M}_u)$: $\mathcal{K}_1 \sim_{\mathcal{R}} \mathcal{K}_2 \Longleftrightarrow \mathcal{K}_1 *_s \mathcal{R} = \mathcal{K}_2 *_s \mathcal{R}$. From Lemma 2 and the fact that $(\mathcal{K} *_s \mathcal{R}) *_s \mathcal{R} = \mathcal{K} *_s \mathcal{R}$ ([3], Proposition 4.2) We infer that any element of the lattice $\mathbb{K}(\mathcal{M}_u)$ Is equivalent to an element of the lattice $\mathbb{K}(\mathcal{I}'')$. Further,

$$\mathcal{K} \subset \mathcal{K} *_s \mathcal{R}$$

So $\mathcal{K} *_s \mathcal{R}$ is the biggest element in its equivalence class $\mathbb{A}(\mathcal{K})$.

LEMMA 3. *Let $\mathbb{A}$ be a class of $\mathcal{R}$-equivalence elements. Then $\mathcal{W}$ is the biggest element in the class $\mathbb{A}$, where $\mathcal{W} = \mathcal{K} *_s \mathcal{R}$ with $\mathcal{K} \in \mathbb{A}$.*□

LEMMA 4. *Let $\mathcal{K}_1 \sim_{\mathcal{R}} \mathcal{K}_2$ and $\mathcal{K}_1 \subset \mathcal{K} \subset \mathcal{K}_2$ be. Then $\mathcal{K}_1 \sim_{\mathcal{R}} \mathcal{K} \sim_{\mathcal{R}} \mathcal{K}_2$.*□

Let $\mathbb{A}$ be a class of $\mathcal{R}$-equivalence elements with the biggest element $\mathcal{W}$. According mentioned above for any element $\mathcal{K} \in \mathbb{A}$ we have

$$rkX \sim rwX, \forall X \in |\mathcal{C}_2\mathcal{V}|.$$

Let be

$$\mathcal{A}' = \cap\{\mathcal{K} | \mathcal{K} \in \mathbb{A}\}.$$

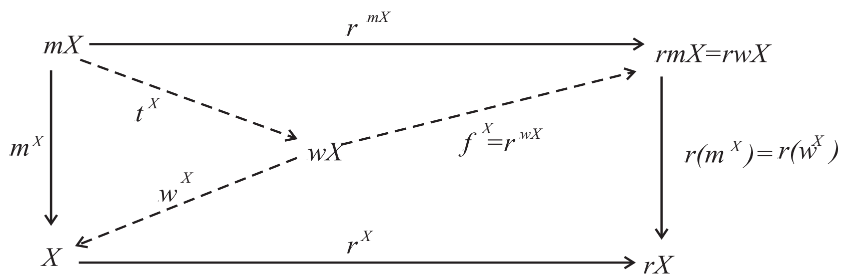
$\mathcal{A}'$ is a coreflective subcategory and because $\tilde{\mathcal{M}} \subset \mathcal{K}$ for any $\mathcal{K} \in \mathbb{A}$, we deduced that $\tilde{\mathcal{M}} \subset \mathcal{A}'$. It is evident, the class $\mathbb{A}$ possesses the smallest element, iff $\mathcal{A}' \in \mathbb{A}$.

LEMMA 5. *Let* $\mathcal{A}' \in \mathbb{A}$ *be. Then* $\mathbb{A} = \{\mathcal{K} \in \mathbb{K}(\mathcal{M}_u) | \mathcal{A}' \subset \mathcal{K} \subset \mathcal{W}\}$, *where* $\mathcal{W} = \mathcal{A}' *_s \mathcal{R}$.□

The following result chows that the smallest element $\overline{\mathcal{M}}$ of the class $\mathbb{K}(\mathcal{I}''(\mathcal{R}))$ can also be obtained as a left product, without resorting to the factorization structure $(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$.

THEOREM 4. $\overline{\mathcal{M}} = \tilde{\mathcal{M}} *_s \mathcal{R}$.

Proof. We build the diagram (PS) for an arbitrary object X of category $\mathcal{C}_2\mathcal{V}$ and for subcategories $\tilde{\mathcal{M}}$ and $\mathcal{R}$.



We write the respectively equality

$$r^X \cdot m^X = r(m^X) \cdot r^{mX}, \tag{1}$$

where $m^X : mX \to X$, is $\tilde{\mathcal{M}}$-coreplica of object X, and r^X and r^{mX} - $\mathcal{R}$-replicas of respective objects and equality (1) takes place for a morphism $r(m^X)$.

$$r^X \cdot w^X = r(m^X) \cdot f^X \tag{2}$$

is the pull-back square built on morphisms r^X and $r(m^X)$. Then

$$m^X = w^X \cdot t^X, \tag{3}$$

$$r^{mX} = f^X \cdot t^X, \tag{4}$$

for a morphism t^X. We have r^X, $m^X \in \mathcal{M}_u$ and from equality (1) it results that $r(m^X) \cdot r^{mX} \in \mathcal{M}_u$.

Because class $\mathcal{M}_u$ is $\mathcal{E}pi$-cohereditary (see [1]) and $r^X \in \mathcal{E}pi$, we deduce that $r(m^X) \in \mathcal{M}_u$. Then in pull-back square (2) it results that $w^X \in \mathcal{M}_u$, and from equality (3) we deduce that $w^X \in \mathcal{E}_u$. So in equality (3) all morphisms belong to the class $\mathcal{E}_u \cap \mathcal{M}_u$. As mentioned above (see [3]) f^X is the $\mathcal{R}$-replica of object wX, and $r(m^X) = r(w^X)$. So square

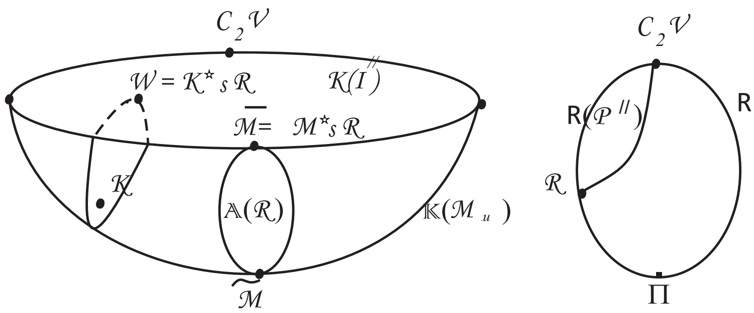
(2) is pull-back, and $w^X \in \mathcal{M}_u$. According to the Theorem 7.3 [4] $w^X \in \mathcal{I}''(\mathcal{R})$. Further, t^X is un epi. Then from equality (4) results that $t^X \in \varepsilon\mathcal{R}$. Because $\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \circ \mathcal{E}_p$ we deduce that $t^X \in \mathcal{P}''(\mathcal{R})$. So we get that (3) is $(\mathcal{P}'', \mathcal{I}'')$-factoring morphism m^X, and the coreflective subcategory $\mathcal{W} = \tilde{\mathcal{M}} *_s \mathcal{R}$ is equal to the subcategory $\overline{\mathcal{M}}$. □

COROLLARY 1. 1. *Class* $\mathbb{A}(\tilde{\mathcal{M}})$ *of elements in* $\mathbb{K}(\mathcal{M}_u)$ $\mathcal{R}$*-equivalents with element* $\tilde{\mathcal{M}}$ *possesses the biggest element* $\overline{\mathcal{M}} = \tilde{\mathcal{M}} *_s \mathcal{R}$, and

$$\mathbb{A}(\tilde{\mathcal{M}}) = \{\mathcal{K} \in \mathbb{K}(\mathcal{M}_u) | \tilde{\mathcal{M}} \subset \mathcal{K} \subset (\mathcal{M} *_s \mathcal{R})\}.$$

2. *Because* $\mathcal{C}_2\mathcal{V} *_s \mathcal{R} = \mathcal{C}_2\mathcal{V}$ *the class of elements* $\mathcal{R}$*-echivalents with* $\mathcal{C}_2\mathcal{V}$ *contains one single element* $\mathcal{C}_2\mathcal{V}$. □

From the previous results, we have the following presentation of the latice $\mathbb{K}(\mathcal{M}_u)$ and the classes of $\mathcal{R}$-equivalence.



Let's highlight how the application $*_s\mathcal{R}$ works on the class $\mathbb{K}(\mathcal{M}_u)$.

Let $\mathcal{R} \in \mathbb{R}$ be. This reflective subcategory generating the factorization structure $(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$. Respectively, this structure divides the lattice $\mathbb{K}$ of nonzero coreflective subcategories into three classes:

$\mathbb{K} - \mathbb{K}(\mathcal{P}'')$, $\mathbb{K}(\mathcal{I}'')$, $\mathbb{K}(\mathcal{P}'', \mathcal{I}'')$.

Remark 1. 1. Always $\mathbb{K}(\mathcal{I}''(\mathcal{R})) \subset \{\mathcal{T} \in \mathbb{K} | \bar{\mathcal{M}} \subset \mathcal{T}\}$. But these classes may be different.

2. $\mathbb{R}(\mathcal{P}''(\mathcal{R})) = \{\mathcal{H} \in \mathbb{R} | \mathcal{R} \subset \mathcal{H}\}$.

3. Let $\mathcal{T}$, $\mathcal{T}_1 \in \mathbb{R}(\mathcal{M}_u)$, $\mathcal{T} \subset \mathcal{T}_1 \subset \mathcal{T} *_s \mathcal{R}$ be. Tnen $\mathcal{T} *_s \mathcal{R} = \mathcal{T}_1 *_s \mathcal{R}$ (see[3]).

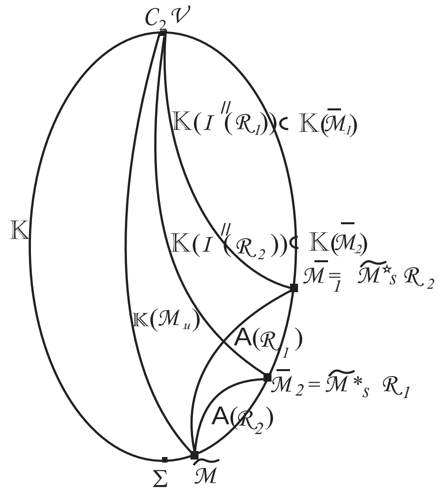
THEOREM 5. *Let* $\mathcal{R}$ *be a reflective subcategory of category* $\mathcal{C}_2\mathcal{V}$, *and* $(\mathcal{P}'', \mathcal{I}'') = (\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$. *Then:*

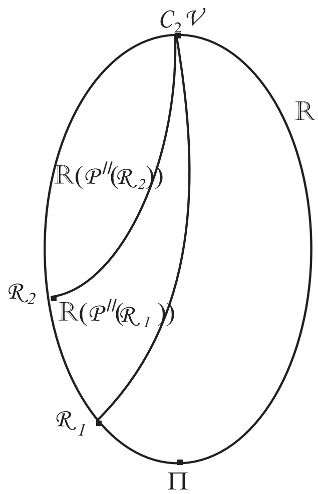
1. $\mathcal{K} *_s \mathcal{R} \in \mathbb{K}(\mathcal{I}'') \Leftrightarrow \mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$.

2. $\mathcal{K} *_s \mathcal{R} = \mathcal{K} \Leftrightarrow \mathcal{K} \in \mathbb{K}(\mathcal{I}'')$.

3. *For any* $\mathcal{K} \in \mathbb{A}(\tilde{\mathcal{M}})$ *and all* $\mathcal{T} \in \mathbb{R}(\mathcal{I}'')$ *it takes place equality* $\mathcal{K} *_s \mathcal{T} = \tilde{\mathcal{M}} *_s \mathcal{R}$.□

Let $\mathcal{R}_1 \subset \mathcal{R}_2$ be two elements of the lattice $\mathbb{R}$. Then $\mathcal{I}''(\mathcal{R}_1) \subset \mathcal{I}''(\mathcal{R}_2)$, and $\mathcal{P}''(\mathcal{R}_2) \subset \mathcal{P}''(\mathcal{R}_1)$. There is the following relationship between these classes.





THEOREM 6. *Let $\mathcal{K} \in \mathbb{K}$, $\mathcal{R} \in \mathbb{R}$ and $\varepsilon\mathcal{R} \subset \mu\mathcal{K}$ be. Then:*

1. $S \subset \mathcal{R}$.
2. *The functor $w : \mathcal{C}_2\mathcal{V} \to \mathcal{K} *_s \mathcal{R}$ is an reflector functor.*
3. *The functors $w : \mathcal{C}_2\mathcal{V} \to \mathcal{K} *_s \mathcal{R}$ and $r : \mathcal{C}_2\mathcal{V} \to \mathcal{R}$ commute: $w \cdot r = r \cdot w$.*

Proof. 1. Because $\mu\mathcal{K} \subset \mathcal{E}_u$ it results that $\varepsilon\mathcal{R} \subset \mathcal{E}_u$, The conditions $\varepsilon\mathcal{R} \subset \mathcal{E}_u$ and $S \subset \mathcal{R}$ are equivalents.

2. It results from Theorem 2.

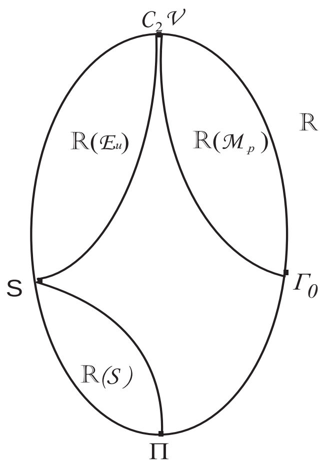
3. We examine the Diagram (PS) for subcategory $(\mathcal{K}, \mathcal{R})$ (see the Diagram from Lemma 2). We have $rwX = rkX$. Further, $krX = kX$, since $\varepsilon\mathcal{R} \subset \mu\mathcal{K}$. Because $r^{rX} = 1$, it follows that $f^{rX} = 1$, or $w^{rX} = 1$. Therefore $wrX = rkrX = rkX$. □

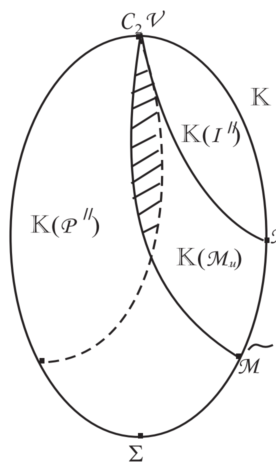
COROLLARY 2. *Either the subcategory $\mathcal{R}$ is $\mathcal{E}_u$-reflective ($S \subset \mathcal{R}$). Then the coreflector functor $\overline{m} : \mathcal{C}_2\mathcal{V} \longrightarrow \overline{\mathcal{M}}$ and the reflector $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $\overline{m} \cdot r = r \cdot \overline{m}$.* □

Analogue takes place transformation of the class $\mathbb{R}(\mathcal{E}_u)$ (class of $\mathcal{E}_u$-reflective subcategories) by the right product.

Example 1. We denote $\mathbb{R}(\mathcal{S}) = \{\mathcal{L} \in \mathbb{R} | \mathcal{L} \subset \mathcal{S}$. Let $\mathcal{L} \in \mathbb{R}(\mathcal{S})\}$ be. Then

1. $\mathcal{I}''(\mathcal{L}) \subset \mathcal{I}''(\mathcal{S}) = \mathcal{M}_p$.
2. $\mathbb{K}(\mathcal{I}''(\mathcal{L})) = \{\mathcal{C}_2\mathcal{V}\}$. $\mathbb{K}(\mathcal{P}''(\mathcal{L})) = \mathbb{K}$. $\mathbb{K}(\mathcal{P}'', \mathcal{I}'') = \{\mathcal{C}_2\mathcal{V}\}$





Example 2. Let $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$ be. Then $\mathcal{P}''(\mathcal{R}) = (\varepsilon\mathcal{R}) \circ \mathcal{E}_p$, where $(\varepsilon\mathcal{R}) \subset \mathcal{M}_p$, and $\mathbb{K}(\mathcal{P}''(\mathcal{R})) = \mathbb{K}(\mathcal{E}_p)$.

Class $\mathbb{K}(\mathcal{E}_p)$ contains the element $\mathcal{C}_2\mathcal{V}$ and with each element contains the bigger elements of the lattice $\mathbb{K}$.

Problem. The class $\mathbb{K}(\mathcal{E}_p)$ contains other elements, except the element $\{\mathcal{C}_2\mathcal{V}\}$?

LEMMA 6. Let $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$ be $(\Gamma_0 \subset \mathcal{R})$. Then $(\varepsilon\mathcal{R}) \perp (\mathcal{E}_u \cap \mathcal{M}_u)$.□

COROLLARY 3. Let $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$ be. Then $\mathbb{K}(\mathcal{I}''(\mathcal{R})) = \{\mathcal{C}_2\mathcal{V}\}$.□

PROPOSITION 2. Let $\mathcal{K} \subset \widetilde{\mathcal{M}}$ and $\mathcal{K} \neq \widetilde{\mathcal{M}}$ be. Then $\mathcal{K} \in \mathbb{K}(\mathcal{P}, \mathcal{I})$.

$$kX \xrightarrow{\quad v_c^X \quad} mX \xrightarrow{\quad m^X \quad} X$$

Proof. Let $m^X : mX \to X$ be the $\tilde{\mathcal{M}}$-coreplica of X, and $v_c^X : kX \to mX$ $\mathcal{K}$-coreplica of mX. There exist an object X so that v_c^X is not isomorphism. So $m^X \cdot v_c^X$ does not belong to the class $\mathcal{M}_u$ and $m^X \cdot v_c^X$ does not belong to the class $\mathcal{E}_p$, because m^X would be an isomorphism.□

Example 3. Let $\mathcal{L} \in \mathbb{R} \backslash (\mathbb{R}(S) \cup \mathbb{R}(\mathcal{M}_p))$ be, and $\mathcal{P}''(\mathcal{L}) = (\varepsilon\mathcal{L}) \circ \mathcal{E}_p$. So $\mathcal{P}''(\mathcal{L})$ intersects with $\mathcal{M}_u$: $\mathcal{P}''(\mathcal{L}) \cap \mathcal{M}_u = \varepsilon\mathcal{L}$. So here it results that $\mathbb{K}(\mathcal{P}''(\mathcal{L})) \cap \mathbb{K}(\mathcal{M}_u)$ may contain other elements except the element $\mathcal{C}_2\mathcal{V}$.

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