Scattered and Digital Topologies in Information Sciences

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Some of the central problems in computer science and, in particular, in programming are the correctness and the similarity problems, which contain:

- the question of whether a program computes a given function;
- the problem to decide whether an element of the space is equal to a fixed element;
- whether two elements of a given space are equal and whether one approximates the other in the specialized order;
- geometric inequalities in information spaces with special distances;
- variational problems in information spaces with special distances;
- geometric control theory in information spaces with special distances;
- the calculation of the weighted means of two strings;

- describe the proper similarity of two strings;
- the problem to calculate the distance between strings;
- the problem to solve the problem of text editing and correction;
- the problem to appreciate changeability of information over time.

A topological space X is called a pseudo-discrete space if the intersection of any family of open sets is open. By definition, the space X is a pseudo-discrete space if and only if the sets $O(x)$, $x \in X$, are open in X. A topological space X is called an Alexandroff space if it is a pseudo-discrete T_0-space. A connected Alexandroff space is called a topological digital space.

We mention the following universal theorem

Theorem 1. *Let $\mathcal{P}$, Γ and $\mathcal{Q}$ be the properties of spaces with the following conditions: any space with property $\mathcal{P}$ has the properties $\mathcal{Q}$ and Γ; a closed subspace of the space with the property Γ is a space with the property Γ; if $Y = Z \cup S$ is a space with the property Γ, where S is a closed subspace with property $\mathcal{P}$ and Z is a subspace with locally property $\mathcal{Q}$ in Y, then the space Y has the property $\mathcal{Q}$; if S and Z are open subspaces of the space Y with the property Γ, F is a subspace of Y with the property $\mathcal{P}$ and $x \in S \setminus Z \subset F$, then there exist an open subset U of Y and a subspace Φ with the property Γ such that $x \in U$, $U \subset \Phi \subset Z \cup (F \setminus Z)$ and $\Phi \setminus Z$*

Then any $\mathcal{P}$-decomposable space X with the property Γ has the property $\mathcal{Q}$.

Theorem 1 opens the possibility of studying $\mathcal{P}$-decomposable spaces using induction and algorithms.

The image classification problem is to find a fragmentation of the image under research into certain regions such that each region represents a class of elementary partitions with the same label. Assume that the domain X of the plane $\mathbb{R}^2$ represented the image of the original Φ and that image is represented by an observed data function $I : X \to \mathbb{R}$ of the level intensity. We have $I(X) = \{c_i : 1 \leq i \leq n\}$. The function I is constructed in the following way: we determine for the image X the levels $\{c_i : 1 \leq i \leq n\} \subset \omega$; find a family $\{O_i : 1 \leq i \leq n\}$ of open subsets of X, where the $O = \cup\{O_i : 1 \leq i \leq n\}$ is dense in X, $O_i \cap O_j = \emptyset$ for $1 \leq i < j \leq n$ and O_i is the set of points of the intensity c_i; for any $i \in \{1, 2, ..., n\}$ and any $x \in O_i$ we put $I(x) = c_i$; if $x \in X \setminus \cup\{O_i : 1 \leq i \leq n\}$, then $I(x) = sup\{i : x \in cl_X O_i\}$; by the method of digitalization we construct a finite subset K of X which represent the image of th original.

On $\mathbb{Z} = \{0, 1, -1, 2, -2, ..., n, -n, ...\}$ one can consider the topology of Khalimsky $\mathcal{T}_{Kh}$ with the open base $\mathcal{B}_{Kh} = \{\{2n-1\} : n \in \mathbb{Z}\} \cup \{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$. The space $(\mathbb{Z}, \mathcal{T}_{Kh})$ is called the Khalimsky line, $(\mathbb{Z}^2, \mathcal{T}^2_{Kh})$ is called the Khalimsky plane, $(\mathbb{Z}^3, \mathcal{T}^3_{Kh})$ is called the Khalimsky space.

Let D be a topological space and $g : D \to \mathbb{Z}$ be a function. For each $n \in \mathbb{Z}$ we put $O(g, n) = \cup\{U \subset X : g(U) = \{n\} : U \textit{ is open in } X\}$. A continuous function f of D in $(\mathbb{Z}, \mathcal{T}_l)$ is a level intensity function on D provided $f(X) = \{0, 1, 2, ..., n\}$ for some $n \in \mathbb{N}$ and $O(f, i) \subset f^{-1}(i) \subset cl_D O(f, i)$ for any $i \in \{0, 1, 2, ..., n\}$. Any intensity function $f : D \to \mathbb{Z}$ determine on D the property $\mathcal{P}(f)$: a subset U of the subspace Y of the space D has the property $\mathcal{P}(f)$ if the set U is open in Y and $f(U)$ is an open singleton subset of $f(Y)$ as the subspace of the space $(\mathbb{Z}, \mathcal{T}_l)$. Relatively to this property D is a $\mathcal{P}(f)$-scattered space.

The space $\mathbb{Z} = \{0, 1, -1, 2, -2, ..., n, -n, ...\}$ is called the discrete line. For the process of digitalization are important the digital topologies on $\mathbb{Z}$. We say that the topology $\mathcal{T}$ on $\mathbb{Z}$ is symmetric if $(\mathbb{Z}, \mathcal{T})$ is a scattered Alexandroff space, the set $\{0\}$ is not open in $(\mathbb{Z}, \mathcal{T})$ and for any $n \in \mathbb{Z}$ the mapping $S_n : \mathbb{Z} \to \mathbb{Z}$, where $S_n(x) = 2n - x$ for each $x \in \mathbb{Z}$, is a homeomorphism.

Theorem 2. *For a topology $\mathcal{T}$ on $\mathbb{Z}$ the following assertions are equivalent:*

1. The topology $\mathcal{T}$ is symmetric.

2. There exists a non-empty subset $L \subset \{2n-1 : n \in \mathbb{N}\}$ such that: $U_0 = \{0\} \cup L \cup \{-n : n \in L\}$ is the minimal open neighbourhood of the point 0 in the space $(\mathbb{Z}, \mathcal{T})$; the family $\mathcal{B}(L) = \{T_{2n}(U_0) : n \in \mathbb{Z}\} \cup \{\{2n-1\} : n \in \mathbb{Z}\}$ is an open base of the topology $\mathcal{T}$ on $\mathbb{Z}$.

The topology of Khalimsky $\mathcal{T}_{Kh}$ with the open base $\mathcal{B}_{Kh} = \{\{2n-1\} : n \in \mathbb{Z}\} \cup \{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$ is of the form $\mathcal{T}(L)$ for $L = \{1\} = L_0$. Therefore the topology of Khalimsky is the unique minimal digital symmetric topology on the discrete line $\mathbb{Z}$.

Recent developments on numerical solutions for hyperbolic systems of conservation laws

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In 1757 Euler developed the famous Euler equations describing the flow of a compressible gas. This is a system of hyperbolic conservation laws in three space dimensions. However until recently one could not show convergence of numerical schemes to the 'classical' weak entropy solutions. By adapting the concept of measure-valued and statistical solutions to multidimensional systems Siddhatha Mishra and his coauthors could recently show convergence of numerical schemes. Mishra has presented these results at the ICM 2018 in Rio de Janeiro. After a brief introduction to the field these developments will be described.