The broken stick model. The optimality ptoperty of the uniform distribution

Gheorghiță Zbăganu

"Gheorghe Mihoc - Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of Romanian Academy, Bucharest, Romania

e-mail:

The problem. Let $I = [0,l]$ be a stick of length l. Let $(X_n)_{1\leq n\leq N+1}$ be iid absolutely continuous random variables valued into I and let F be their distribution. Let $(X_{n:N})_{1\leq n\leq N+1}$ be their order statistic and $Y_n = X_{n:N} - X_{(n-1):N}, 1 \leq n \leq N+1$ be the corresponding spacings. Here $X_{0:N} = 0$. Let $G_n = G_n(F) = \frac{1}{N}\sum \delta_{\frac{NY_n}{l}}$ be the normalized empirical distributions of the spacings. It is known that if F is the uniform distribution, then G_n weakly converges to the exponential distribution $Exp(1)$ hence the Lorenz curves $L_N(x) = \int_0^x G_n^{-1}(t)\,dt$ converge to the Lorenz curve of the exponential distribution $L(x) = x + (1-z)\ln(1-x)$. See for example *Towards understanding the Lorenz curve using the Uniform distribution* ,Chris J. Stephens,Gini-Lorenz Conference, University of Siena, Italy, May 2005).
Results. Let us say that a distribution on an interval I has the property (D) if the distributions $G_n(F)$ weakly converge to some limit distribbution $H = H(F)$. We prove

Theorem 1. *Let $a_0 < a_1 < a_2 < ... < a_k$ and $I_j = [a_{j-1}, a_j)$. Suppose that $F = \sum_{j=1}^{k} p_j F_j$ where*

F_j are distributions on I_j having the property (D) and $p_j > 0, \sum_{j=1}^{k} p_j = 1$.

Then F has the property (D) ,too. Precisely, if $\lim G_n(F_j) = H_j$ then $\lim G_n(F) = \sum_{j=1}^{k} p_j H_j \circ h^{-1}_{\frac{\pi_j}{p_j}}$

where $\pi_j = \frac{a_j - a_{j-1}}{a_k - a_0}$ and $h_\alpha(x) = \alpha x$ is the homothethy.

Corollary 1. *Suppose that F_j =Uniform(I_j) . Then $H = \sum_{j=1}^{k} p_j Exp\left(\frac{p_j}{\pi_j}\right)$*

Corollary 2. *COROLLARY 3. If F has a density f of the form* $f(x) = \sum_{j=1}^{k} \alpha_j 1_{I_j}$ *then the Lorenz curve of H is under the graph of* $x + (1-x)\ln x$

Conclusion. The uniform distribution is the most egalitarian.