On the semigroup of endomorphisms of a topological universal algebra

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Let $\mathbb{N} = \{1, 2, ...\}$ be the set of natural numbers and $n \in \omega = \{0, 1, 2, ...\}$ be the set of non-negative integers. The discrete sum $\Omega = \oplus\{\Omega_n : n \in N = \{0, 1, 2, ...\}\}$ of the pairwise disjoint topological spaces $\{\Omega_n : n \in N\}$ is called a continuous signature. If the space Ω is a discrete space, then we say that Ω is a discrete signature.

A topological Ω-algebra or a topological universal algebra of the signature Ω is a family $\{G, e_{nG} : n \in N\}$, where G is a non-empty topological space and $e_{nG} : \Omega_n \times G^n \to G$ is a continuous mapping for each $n \in \omega$. The concept of universal algebra was created by Alfred North Whitehead in 1898 as a generalization of Boole's logical algebras. The term universal algebra was proposed by James Joseph Sylvester [9]. Between 1935 and 1950 important works were published by Garrett Birkhoff [1, 2]. As in [4, 5, 7, 8] we continue the study of semigroups of endomorphisms of universal topological algebras.

Let A, B and C be three topological universal algebras of signature Ω. The function $f : A \longrightarrow B$ is called a morphism or homomorphism, if $f(u(x)) = u(f^n(x))$ for any $n \in \omega$, any $u \in \Omega_n$ and any element $x = (x_1, x_2, ..., x_n) \in G^n$, where $f^n(x) = (f(x_1), f(x_2), ..., f(x_n))$. The composition of the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is the function $h = f \cdot g$: $A \longrightarrow C$, where $h(x) = g(f(x))$ for any $x \in A$. The composition of two continuous morphisms is always a continuous morphism. A morphism that is a bijective function is called an isomorphism. An isomorphism which is a homeomorphisms called a topological isomorphism.

If a topological isomorphism can be established between two topological universal algebras, they are called topologic isomorphs. Two topological isomorphs topological universal algebras are identified. Morphisms, respectively isomorphisms, between a topological universal algebra and itself are called endomorphisms, respectively automorphisms.

A semigroup S equipped with a topology is called a semi-topological semigroup if the tranlations $\{u_a, \varphi_b : a, b \in G\}$, where $u_a(x)a \cdot x$ and $\varphi_b(x)$ for all $a, b, x \in S$, are continuous mappings of the space S into itself.

The family of all continuous endomorphisms $End_c(G)$ of a topological universal algebra G relatively to the operation of composition $f \cdot g$ is a semigroup with the unity. The semigroup $End_c(G)$ in the topology of pointwise convergence is a semi-topological semigroup.

Let Ω be a fixed signature. A topological universal algebra G is a topological free universal algebra in some class of universal algebras if there is given a subspace $I = I_G \subset G$ with the properties:

1) the algebra G is generate by the set I, i.e. $G= s_G(I)$, and I is called the space of generators of G;

2) for any continuous mapping $f : I \longrightarrow G$ there exists a (unique) continuous endomorphism $\hat{f} : G \longrightarrow G$ such that $f(x) = \hat{f}(x)$ for each $x \in I$.

A universal algebra A is called cyclic if there exists a point $a \in G$ such that the set $\{a\}$ generate the algebra G.

G. Gratzer and E. T. Schmidt [3] proved that any complete lattice is isomorphic to the lattice of congruence of some universal algebra.

The following theorem is a generalization and conceptualization of the theorem from ([8], pag.98).

Theorem 1. *For any semi-topological semigroup with unity S there exist a discrete signature Ω and a topological universal algebra G_S of signature Ω such that the semi-topological semigroups S and $End(G_S)$ are topological isomorphic.*

Theorem 2. *For any topological semigroup with unity S there exist a continuous signature Ω and a topological universal algebra G_S of signature Ω such that the topological semigroups S and $End(G_S)$ are topological isomorphic.*

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