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LIFETIME DISTRIBUTIONS AND THEIR APPROXIMATION IN RELIABILITY OF SERIAL/PARALLEL NETWORKS

Alexei LEAHU, Veronica ANDRIEVSCHI-BAGRIN

Abstract: In this paper we present limit theorems for lifetime distributions connected with network's reliability as distributions of random variables(r.v.) $\min(Y_1,Y_2,...,Y_M)$ and $\max(Y_1,Y_2,...,Y_M)$, where $Y_1,Y_2,...$, are independent, identically distributed random variables (i.i.d.r.v.), M being Power Series Distributed (PSD) r.v. independent of them and, at the same time, Y_k , $k = 1, 2, ...$, being a sum of non-negative, i.i.d.r.v. in a Pascal distributed random number.

1. Introduction

First of all, let us observe that many mathematical models connected with Network's Reliability deal with series and parallel Networks as subsystems of Network with more complex structure/topology. At the same time, the lifetime of each element of the Network, as r.v., may be represented as a sum of fixed or of random number of nonnegative r.v.[4]. Finally, reliability characteristics of such Networks depend of the same characteristics of such kind of Network subsystems or elements. This explains the appearance in the last decades of new probability distributions of the lifetime as a function of the minimum/maximum or sum of nonnegative r.v. An entire class of such distributions can be described as Min / Max Power Series Distributions [6] and PSD Convolutions [8]. If we refer to the above mentioned results, in the

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context of mathematical modeling of Network's lifetime,then the Min/Max distributions target the case when the units/elements of the Network are non-replaceable in the event of their fall. The purpose of our work is to extend these results to the case when each unit of the network can be instantly replaced by a similar standby unit. Because we refer to the above named distributions, let us remember the main notions and results which will be used. Before that, we mention the following: the introduction and use of the $0-$truncation procedure of discrete-type distributions describing the random number of r.v. which are listed in the Min / Max or random sum of v.a., all contextually linked to the notion of lifetime, are motivated by the fact that real Networks consist of at least one unit or element

2. Power series distributions and related results

Introduction of the "power series distribution" class is due to Noack [10], highlighting a subset of discrete probabilistic distributions [3] such as binomial, Poisson, logarithmic, geometric, negative binomial, Pascal distributions and many others.

Definition 1. We say that *Z is a Power Series Distributed r.v. with parameter θ and power series function $A(\theta)=\sum\limits_{z\geqslant 0} a_z\theta^z$, shortly $Z\in PSD$,* if

$$\mathbf{P}(Z=z)=\frac{a_z\theta^z}{A(\theta)},\ a_z\geqslant 0,\ z=0,1,2,...;\theta\in(0,\tau),$$

where the power series $\sum\limits_{z\geqslant 1} a_z\theta^z$ is convergent with radius of convergence $\tau\in(0,+\infty)\cup\{+\infty\}$.

Since the PSD discrete probability distributions used in our paper will be the 0-truncated ones, we must be sure that this operation does not alter the quality of the distribution to be of PSD class.

Proposition 1. *If the r.v. $Z\in PSD$ with parameter $\theta\in(0,\tau)$, $\tau\in(0,+\infty)\cup\{+\infty\}$ and power series function $A(\theta)=\sum\limits_{z\geqslant 0} a_z\theta^z$, then its 0-truncation is a r.v. $Z^*\in PSD$ with parameter $\theta\in(0,\tau)$, $\tau\in(0,+\infty)\cup\{+\infty\}$ and power series function $A^*(\theta)=\sum\limits_{z\geqslant 1} a_z\theta^k=A(\theta)-a_0$, i.e.,*

$$\mathbf{P}(Z^*=z)=\frac{a_z\theta^z}{A^*(\theta)}=\frac{a_z\theta^z}{A(\theta)-a_0},\ a_z\geqslant 0,\ z=1,2,...\ .$$

Proof. Let us consider the r.v. $Z\in PSD$ with parameter $\theta\in(0,\tau)$, $\tau\in(0,+\infty)\cup\{+\infty\}$ and power series function $A(\theta)=\sum\limits_{k\geqslant 0} a_k\theta^k$. Then its

0-truncation will be the discrete r.v. Z^* with distribution

$$\mathbf{P}(Z^* = z) = \mathbf{P}(Z = z / Z \geqslant 1) = \frac{a_z\theta^z}{A(\theta)} / (1 - \frac{a_0}{A(\theta)}) = \frac{a_z\theta^z}{A(\theta) - a_0},$$

where $\theta \in (0, \tau)$, $\tau \in (0, +\infty) \cup \{+\infty\}.a_z \geqslant 0$, $z = 1,2,...$, excluding the degenerate case $A(\theta) - a_0 = 0$ when 0-truncation does not make sense. □

Remark. If the r.v. $Z \in PSD$ and the null coefficient a_0 of its power series function $A(\theta) = \sum\limits_{z \geqslant 0} a_z\theta^z$ is equal to 0, then the 0-truncation of the r.v. Z does not change the initial distribution.

Example 1. The following Table 1 shows the form of PSD parameters of 0-truncated distributions of some classical discrete distributions as $Bin(n;p)$, $Geom(p)$, $Poisson(\lambda)$, $Log(p)$, $NegBin(k;p)$, $Pascal(k;p)$, marked by symbol " $*$ " , if their 0-truncation changes form as a PSD.

Here, according to the [3],the $Pascal(k;p)$ distribution expresses the probability of having to wait exactly z Bernoulli trials until k "successes" have occurred if the probability of a success in a single trial is p (probability of "failure" $q = 1 - p$). At the same time, the Negative Binomial distribution ($NegBin(k;p)$) expresses the probability of the number z of all "failures" occurring while waiting for k "successes" in Bernoulli trials.

Distribution	a_z	θ	$A(\theta)$	τ
$Bin^*(n;p)$, $n \in \{1,2,..\}$, $0 < p < 1$	$\begin{cases} \binom{n}{z}, for\ z = \overline{1,n}, \\ 0,\ for\ z = 0\ or\ z > n. \end{cases}$	$\frac{p}{1-p}$	$(1+\theta)^n - 1$	$+\infty$
$Poisson^*(\lambda)$, $\lambda > 0$	$\begin{cases} \frac{1}{z!}, for\ z = 1,2,.., \\ 0,\ for\ z = 0. \end{cases}$	λ	$e^\theta - 1$	$+\infty$
$Log(p)$, $0 < p < 1$	$\begin{cases} \frac{1}{z}, for\ z = 1,2,.., \\ 0,\ for\ z = 0. \end{cases}$	p	$-\ln(1-\theta)$	1
$Geom^*(p)$, $0 < p < 1$	$\begin{cases} 1, for\ z = 1,2,.., \\ 0,\ for\ z = 0. \end{cases}$	$1 - p$	$\frac{\theta}{1-\theta}$	1
$NegBin^*(k;p)$, $k \in \{1,2,..\}$, $0 < p < 1$	$\begin{cases} \binom{z+k-1}{z}, for\ z = 1,2,.., \\ 0,\ for\ z = 0. \end{cases}$	p	$(1-\theta)^{-k} - 1$	1
$Pascal(k;p)$, $k \in \{1,2,..\}$, $0 < p < 1$	$\begin{cases} \binom{z-1}{k-1}, for\ z = k, k+1,.., \\ 0,\ for\ z = \overline{0, k-1}. \end{cases}$	$1 - p$	$\left(\frac{\theta}{1-\theta}\right)^k$	1

Table 1.

Next results refers to the distributions of PSD mixtures of minimum or maximum of nonnegative i.i.d.r.v.

Proposition 2 [6]. *If* X_1, X_2,...,X_n,... *are nonnegative i.i.d.r.v. with cumulative distribution functions (c.d.f.)* $F(x) = \mathbf{P}(X_i \leq x)$, $i \geq 1$ *and the r.v.* $N \in PSD$ *with parameter* $\theta \in (0, \tau)$, $\tau \in (0, +\infty) \cup \{+\infty\}$ *whose power series function* $A(\theta) = \sum_{k \geqslant 0} a_k \theta^k$, N *being independent of r.v.* X_1, X_2,...,X_n,..., *then the c.d.f. of r.v.* $U_N = \min(X_1, X_2,...,X_N)$ *and* $V_N = \max(X_1, X_2,...,X_N)$ *are given, respectively, by formulas*

$$U_N(x) = \mathbf{P}(U_N \leq x) = 1 - \frac{A(\theta(1 - F(x))}{A(\theta)} I_{[0,+\infty)}(x),$$

$$V_N(x) = \mathbf{P}(V_N \leq x) = \frac{A(\theta F(x))}{A(\theta)} I_{[0,+\infty)}(x),$$

where $I_{[0,+\infty)}(x) = \begin{cases} 0, & \textit{if } x < 0, \\ 1, & \textit{if } x \geq 0. \end{cases}$

Lifetime distributions generated by formulas from Proposition 2 will be called *lifetime distributions of Min-PSD type* and, respectively of *Max-PSD type*. Distributions that fit into these types of lifetime distributions have been proposed and studied, for example, in the papers of Adamidis [1], Kus [5], and many others authors. The idea that has led us to describe these two classes of distribution is very close to the approach made in the paper [7]. The only difference is that we focused on 0−truncated distributions.

Another type of lifetime distribution gives us the distribution of a random sum of non-negative, i.i.d.r.v. This kind of lifetime distribution appears when the unit or element of the Network consists, for example, of $k \geq 1$ pairs of identical interchangeable units. Initially the first pair of units operates, one of the units being in standby mode, so that the defective unit, which immediately goes through the repair and standby mode, can be replaced instantaneously with the waiting unit with the probability $1 - p, p \in (0, 1)$, i.e., with the probability p this unit cannot be replaced by the standby unit and this pair of units fails. Suppose that if a particular pair of units fails, then it is replaced by the next waiting pair. So the whole element fails when the last pair fails. Each pair of unities has the same lifetime c.d.f., so we deduce that life time of pair no. j is a r.v. $Y_{N_j} \sim X_1 + X_2 + ... + X_{N_j}$, where X_1, X_2,... are i.i.d.r.v. which corresponds to the lifetime of each unit and N_j are i.i.d.r.v., $N_j \sim Geom^*(p)$, N_j being independent of r.v. X_1, X_2,..., $j = \overline{1, k}$. This means that the lifetime of the described element is a r.v. $Y_N \sim Y_{N1} + Y_{N2} + ... + Y_{Nk} \sim X_1 + ... + X_N$, where $N \sim Pascal(k, p)$, $k \in \{1, 2, ...\}$, $p \in (0, 1)$.

Unfortunately, the exact distribution of the lifetime as a distribution of the sum of non-negative, independent, identically distributed random variables taken in a PSD random number can, generally, be determined in terms of the Probability Generating Function (in discrete case) or in the terms of Laplace transformation (in absolutely continuous case).

Proposition 3 [9]. *If X_1, X_2,...,X_n,... are nonnegative i.i.d.r.v. and r.v. $N \in PSD$ with parameter $\theta \in (0, \tau)$, $\tau \in (0, +\infty) \cup \{+\infty\}$ and the power series function $A(\theta) = \sum_{k \geqslant 0} a_k \theta^k$, N being independent of the r.v. X_1, X_2,...,X_n,..., then:*

a) the lifetime $Y_N = X_1 + ... + X_N$ is a discrete r.v. with probability distribution given by its Probability Generating Function

$$\psi_{Y_N}(w) = \frac{A(\theta\psi(w))}{A(\theta)},$$

as soon as the probability distributions of the r.v. X_i, $i \geq 1$, are given by their Probability Generating Function $\psi(w)$;

b) the lifetime $Y_N = X_1 + ... + X_N$ is an absolutely continuous r.v. with probability density function (p.d.f.) $f_{Y_N}(x)$ given by Laplace Transform

$$\varphi_{Y_N}(s) = \int_0^{+\infty} e^{-sx} f_{Y_N}(x) dx = \frac{A(\theta\varphi(s))}{A(\theta)},$$

as soon as X_i, $i \geq 1$, are r.v. with p.d.f. given by their Laplace Transform $\varphi(s)$.

Even if it is difficult to deduce explicitly lifetime distribution from the previous sentence, it is easy to calculate lifetime's mean value and variance due to

Consequence. *In the conditions of Proposition 4 the following formulas are valid:*

$$\mathbb{E}Y_N = \mathbb{E}N \cdot \mathbb{E}X_1, \textit{ which coincides with Wald's identity,}$$

$$Var(Y_N) = (\mathbb{E}X_1)^2 \cdot Var(N) + \mathbb{E}N \cdot Var(X_1).$$

Example 2. From the Proposition 3 we have that if

a) $X_i \sim Geom^*(p^*)$, $p^* \in (0, 1)$, $i \geq 1$, and $N \sim Pascal(k; p)$, $k \geq 1$, $p \in (0, 1)$, then the lifetime $Y_N \sim Pascal(k, pp^*)$;

b) $X_i \sim Exp(\lambda)$, $\lambda > 0$, $i \geq 1$, and $N \sim Pascal(k; p)$, $k \geq 1$, $p \in (0, 1)$, then the lifetime $Y_N \sim Erlang(k, \lambda p)$.

Note that in most cases to find lifetime's distribution on the base of Proposition 3 numerical methods are required for inversion of the Probability Generating Function or Laplace Transform. But, in some special cases, the lifetime distribution may be approximated by its limit distribution, using, for example, Brown's Limit Theorem [2] or its expanded version:

Theorem [8]**.** *If* $Y_N = X_1+X_2+...+X_N$*, where* $(X_n)_{n\geq 1}$ *are nonnegative i.i.d.r.v. governed by the Strong Law of Large Numbers, i.e.,*

$$\mathbf{P}\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n} X_i \ = \frac{1}{\lambda}\right) = 1,\ \lambda > 0$$

and $N \sim Pascal(k;p)$*,* $k \in \{1,2,...\}$*,* $p \in (0,1)$*, r.v.* N *being independent of r.v.* $(X_n)_{n\geq 1}^{;}$*, then we have the following convergence in distribution:* $\lambda p Y_N \underset{p\longrightarrow 0}{\Longrightarrow} Y$*, where the limit r.v.* $Y \sim Erlang(k;1)$*.*

Consequence [8] **.** *In the conditions of the above Theorem, for very small values of parameter* p*,* $p \in (0,1)$*, the c.d.f.*

$$F_{Y_N}(x) = \mathbf{P}(Y_N \leq x) \simeq F_Y(x) = \mathbf{P}(Y \leq x) = 1 - e^{-\lambda p x}\sum_{j=0}^{k-1}\frac{(\lambda p x)^j}{j!}.$$

3. Approximations for some Min-PSD and Max-PSD lifetime distributions.

Based on the above presented results, we can address the issue of the approximation of Min-PSD and Max-PSD lifetime distributions, taking into account that the lifetime distributions of the component elements are not explicitly known.

Proposition 4. *If* $Y_1, Y_2, ...Y_i, ...$*are i.i.d.r.v., where* $Y_i \sim X_1 + X_2 + ... + X_N$, for each $i \geq 1$ *is a random sum on the base of the sequence* $(X_n)_{n\geq 1}$ *of nonnegative i.i.d.r.v. governed by the Strong Law of Large Numbers, i.e.,*

$$\mathbf{P}\left(\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n} X_i \ = \frac{1}{\lambda}\right) = 1,\ \lambda > 0$$

and $N \sim Pascal(k;p)$*,* $k \in \{1,2,...\}$*,* $p \in (0,1)$*, the r.v.* N *being independent of r.v.* $(X_n)_{n\geq 1}^{;}$*, then we have the following convergence in distribution of r.v.*

$$\lambda p U_m(p) = \lambda p \min(Y_1, Y_2, ..., Y_m) \underset{p\to 0}{\Longrightarrow} U_m,$$

$$\lambda p V_m(p) = \lambda p \max(Y_1, Y_2, ..., Y_m) \underset{p\to 0}{\Longrightarrow} V_m,$$

where the c.d.f.

$$F_{U_m}(x) = \mathbf{P}(U_m \le x) = \left[1 - e^{-mx} \left(\sum_{j=0}^{k-1} \frac{x^j}{j!}\right)^m\right] I_{[0,+\infty)}(x),$$

$$F_{V_m}(x) = \mathbf{P}(V_m \le x) = \left(1 - e^{-x} \sum_{j=0}^{k-1} \frac{x^j}{j!}\right)^m I_{[0,+\infty)}(x).$$

Proof. Because $\lambda p U_m(p) = \lambda p \min(Y_1, Y_2, ...Y_m) = \min(\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m)$ and $\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m$ are i.i.d.r.v. we have that

$$\mathbf{P}(\lambda p U_m(p) \le x) = \mathbf{P}(\min(\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m) \le x) =$$
$$= 1 - [1 - \mathbf{P}(\lambda p Y_1 \le x)]^m .$$

In the same way

$$\mathbf{P}(\lambda p V_m(p) \le x) = \mathbf{P}(\max(\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m) \le x) =$$
$$[\mathbf{P}(\lambda p Y_1 \le x)]^m .$$

But $\min(\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m)$ and $\max(\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m)$ are continuous functions of r.v. $\lambda p Y_1, \lambda p Y_2, ..., \lambda p Y_m$ which, according to the previous Theorem [8], weakly converges to the same Erlang distribution $Erlang(k;1)$. So, using the expression of Erlang distribution $Erlang(k;1)$ and the above expressions for $\mathbf{P}(\lambda p U_m(p) \le x)$, $\mathbf{P}(\lambda p V_m(p) \le x)$, we deduce explicitly their limit distributions when $p \to 0$. $\square$

Consequence 1. *In the conditions of Proposition 4, for the small values of parameter p, $p \in (0,1)$, we have the following formulas of approximations for the lifetime's c.d.f.*

$$F_{U_m(p)}(x) = \mathbf{P}(U_m(p) \le x) \simeq \left[1 - e^{-\lambda p m x} \left(\sum_{j=0}^{k-1} \frac{(\lambda p x)^j}{j!}\right)^m\right] I_{[0,+\infty)}(x),$$

$$F_{V_m(p)}(x) = \mathbf{P}(V_m(p) \le x) \simeq \left(1 - e^{-\lambda p x} \sum_{j=0}^{k-1} \frac{(\lambda p x)^j}{j!}\right)^m I_{[0,+\infty)}(x).$$

Proof. Proposition 4 shows, in fact, that for the small values of parameter p, $p \in (0,1)$ we have the following formulas of approximations for the lifetime's c.d.f.

$$F_{\lambda p U_m(p)}(x) = \mathbf{P}(\lambda p U_m(p) \le x) \simeq \left[1 - e^{-mx} \left(\sum_{j=0}^{k-1} \frac{x^j}{j!}\right)^m\right] I_{[0,+\infty)}(x),$$

$$F_{\lambda p V_m(p)}(x) = \mathbf{P}(\lambda p V_m(p) \le x) \simeq \left(1 - e^{-x} \sum_{j=0}^{k-1} \frac{x^j}{j!}\right)^m I_{[0,+\infty)}(x).$$

But these formulas are equivalent to the formulas from our consequence because, for example,

$$F_{U_m(p)}(x) = \mathbf{P}(\lambda p U_m(p) \le \lambda p x) \simeq \left[1 - e^{-m\lambda p x} \left(\sum_{j=0}^{k-1} \frac{(\lambda p x)^j}{j!}\right)^m\right] I_{[0,+\infty)}(x). \ \square$$

The fact that the 0−truncated geometrical distribution with parameter $p \in (0,1)$ coincides with the Pascal distribution $Pascal(1;p)$ implies the following

Consequence 2. *In conditions of Proposition 4, if* $N \sim Geom^*(p)$, $p \in (0,1)$, *then*

$$\lambda p U_m(p) = \lambda p \min(Y_1, Y_2, ..., Y_m) \underset{p \to 0}{\Longrightarrow} U_m,$$

$$\lambda p V_m(p) = \lambda p \max(Y_1, Y_2, ..., Y_m) \underset{p \to 0}{\Longrightarrow} V_m,$$

where the c.d.f.

$$F_{U_m}(x) = \mathbf{P}(U_m \le x) = (1 - e^{-mx})\ I_{[0,+\infty)}(x),$$

$$F_{V_m}(x) = \mathbf{P}(V_m \le x) = (1 - e^{-x})^m I_{[0,+\infty)}(x).$$

This means that, for very small values of parameter p, $p \in (0,1)$, *we have the following formulas of approximations for the lifetime's c.d.f.*

$$F_{U_m(p)}(x) = \mathbf{P}(U_m(p) \le x) \simeq \left[1 - e^{-\lambda p m x}\ \right] I_{[0,+\infty)}(x),$$

$$F_{V_m(p)}(x) = \mathbf{P}(V_m(p) \le x) \simeq \left(1 - e^{-\lambda p x}\ \right)^m I_{[0,+\infty)}(x).$$

If in Proposition 4 we will give up the supposition that the number m is fixed, by replacing it with a random number $M \in PSD$, then we may formulate its extension as a

Proposition 5. *If* $Y_1, Y_2, ...Y_i, ...$ *are i.i.d.r.v., where* $Y_i \sim X_1 + X_2 + ... + X_N$ *for each* $i \ge 1$ *is a sum on the base of the sequence* $(X_n)_{n \ge 1}$ *of nonnegative i.i.d.r.v. governed by the Strong Law of Large Numbers, i.e.,*

$$\mathbf{P}\left(\lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \ = \frac{1}{\lambda}\right) = 1,\ \lambda > 0$$

and the the r.v. $N \sim Pascal(k;p)$, $k \in \{1, 2, ...\}$, $p \in (0,1)$, N *being independent of r.v.* $(X_n)_{n \ge 1}$, *then for each random number* $M \in PSD$ with parameter $\theta \in (0,\tau)$, $\tau \in (0,+\infty) \cup \{+\infty\}$, *and its power series function* $A(\theta) = \sum\limits_{k\geqslant 0} a_k\theta^k$, *$M$ being independent of r.v.* $(Y_m)_{m\geq 1}$, *we have the following convergence in distribution of r.v.*

$$\lambda p U_M(p) = \lambda p \min(Y_1, Y_2, ..., Y_M) \underset{p\to 0}{\Longrightarrow} U,$$

$$\lambda p V_M(p) = \lambda p \max(Y_1, Y_2, ..., Y_M) \underset{p\to 0}{\Longrightarrow} V,$$

where the c.d.f.

$$F_U(x) = \mathbf{P}(U \leq x) = \left[1 - A\left(\theta e^{-x}\sum_{j=0}^{k-1}\frac{x^j}{j!}\right) \diagup A(\theta)\right] I_{[0,+\infty)}(x),$$

$$F_V(x) = \mathbf{P}(V \leq x) = \left[A\left(\theta\left(1 - e^{-x}\sum_{j=0}^{k-1}\frac{x^j}{j!}\right)\right) \diagup A(\theta)\right] I_{[0,+\infty)}(x).$$

Proof. Let's observe that, from Formula of Total Probability, we have

$$\mathbf{P}(\lambda p U_M(p) \leq x) = \mathbf{P}(\lambda p \min(Y_1, Y_2, ..., Y_M) \leq x) =$$

$$\sum_{m\geq 1}\mathbf{P}(\{\lambda p \min(Y_1, Y_2, ..., Y_M) \leq x\} \diagup \{M = m\}) \cdot\ \mathbf{P}(M = m) =$$

$$\sum_{m\geq 1}\mathbf{P}(\lambda p \min(Y_1, Y_2, ..., Y_m) \leq x) \cdot \frac{a_m\theta^m}{A(\theta)}.$$

But, according Proposition 5, $\lambda p \min(Y_1, Y_2, ..., Y_m) \underset{p\to 0}{\Longrightarrow} U_m$, where

$$F_{U_m}(x) = \mathbf{P}(\lambda p \min(Y_1, Y_2, ..., Y_m) \leq x) = \left[1 - \left(e^{-x}\sum_{j=0}^{k-1}\frac{x^j}{j!}\right)^m\right] I_{[0,+\infty)}(x).$$

So, $\lambda p \min(Y_1, Y_2, ..., Y_M) \underset{p\to 0}{\Longrightarrow} U$, where

$$F_U(x) = \mathbf{P}(U \leq x) = \sum_{m\geq 1}\left[1 - \left(e^{-x}\sum_{j=0}^{k-1}\frac{x^j}{j!}\right)^m\right] I_{[0,+\infty)}(x) \cdot \frac{a_m\theta^m}{A(\theta)} =$$

$$= \left[1 - A\left(\theta e^{-x}\sum_{j=0}^{k-1}\frac{x^j}{j!}\right) \diagup A(\theta)\right] I_{[0,+\infty)}(x).$$

In the same way, we may prove the second part of our Proposition. □

Consequence 1. *In the conditions of Proposition 5, for very small values of parameter p, $p \in (0,1)$, we have the following formulas of approximations for the lifetime's c.d.f.*

$$F_{U_M(p)}(x) = \mathbf{P}(U_M(p) \le x) \simeq \left[1 - A\left(\theta e^{-\lambda px} \sum_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}\right) \diagup A(\theta)\right] I_{[0,+\infty)}(x),$$

$$F_{V_M(p)}(x) = \mathbf{P}(V_M(p) \le x) \simeq \left[A\left(\theta\left(1 - e^{-\lambda px} \sum_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}\right)\right) \diagup A(\theta)\right] I_{[0,+\infty)}(x).$$

Now, using the PSD form of the $0-$truncated geometrical distribution and the past Consequence 1, we have the

Consequence 2. *In the conditions of Proposition 5, for very small values of parameter p, $p \in (0,1)$, if the r.v. $M \sim Geom^*(p^*)$, $p^* \in (0,1)$,* then *we have the following formulas of approximations for the lifetime's c.d.f.*

$$F_{U_M(p)}(x) = \mathbf{P}(U_M(p) \le x) \simeq \left[\frac{1 - e^{-\lambda px} \sum\limits_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}}{1 - (1-p^*) e^{-\lambda px} \sum\limits_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}}\right] I_{[0,+\infty)}(x),$$

$$F_{V_M(p)}(x) = \mathbf{P}(V_M(p) \le x) \simeq \frac{p^*\left(1 - e^{-\lambda px} \sum\limits_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}\right)}{1 - (1-p^*)\left(1 - e^{-\lambda px} \sum\limits_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}\right)} I_{[0,+\infty)}(x).$$

Conclusion. Comparing the c.d.f. $F_{U_M(p)}(x)$ and $F_{V_M(p)}(x)$ which describes the probabilistic behavior of lifetime, respectively for serial and parallel Networks with replaceable units and random number of units, we may deduce, for example, from Consequence 2, that $F_{U_M(p)}(x) \geqslant F_{V_M(p)}(x)$ for every real x. Indeed, to verify that for $x \in [0,+\infty)$, it is sufficient to verify that the difference

$$\frac{1}{1 - (1-p^*) e^{-\lambda px} \sum\limits_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}} - \frac{p^*}{1 - (1-p^*)\left(1 - e^{-\lambda px} \sum\limits_{j=0}^{k-1} \frac{(\lambda px)^j}{j!}\right)} =$$

$$= \frac{(1-p^*)(1+p^*)e^{-\lambda px}\sum_{j=0}^{k-1}\frac{(\lambda px)^j}{j!}}{\left[1-(1-p^*)e^{-\lambda px}\sum_{j=0}^{k-1}\frac{(\lambda px)^j}{j!}\right]\left[1-(1-p^*)\left(1-e^{-\lambda px}\sum_{j=0}^{k-1}\frac{(\lambda px)^j}{j!}\right)\right]} \geqslant 0.$$

This is true because $p^* \in (0,1)$ and all the functions appearing in the last fraction are non-negatives. So, survival/reliability functions $S_{U_M(p)}(x) = 1 - F_{U_M(p)}(x) \leqslant S_{V_M(p)}(x) = 1 - F_{V_M(p)}(x)$. In other words, in our circumstances, the reliability of parallel Networks is higher than the reliability of serial Networks. By the way, this corresponds to the common sense considerations.

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Alexei LEAHU,
Technical University of Moldova
Email: alexei.leahu@ati.utm.md

Veronica ANDRIEVSCHI-BAGRIN,
Technical University of Moldova
Email: veronica.bagrin@ati.utm.md