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On skew polynomial rings and some related rings

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Abstract

For a ring A with identity and a monoid G we consider "monoid rings" with respect to G over A where the multiplication $(a \cdot x)(b \cdot y)$ $(a, b \in A,\ x, y \in G)$ is determined by a monoid homomorphism $G \to End(A)$. Examples include various skew polynomial rings. There is also a link to $\mathbb{Z}_2-$ graded rings.

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A system called a D-structure in [3] and introduced in [2] consists of a ring A with identity 1, a monoid G with identity e and mappings $\sigma_{x,y} : A \to A$ for $x,\ y \in G$ satisfying the following condition:

Condition (A)

(0) For each $x \in G$ and $a \in A$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

(i) Each $\sigma_{x,y}$ is an additive endomorphism.

(ii) $\sigma_{x,y}(ab) = \sum_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

(iv$_1$) $\sigma_{x,y}(1) = 0$ if $x \neq y$; (iv$_2$) $\sigma_{x,x}(1) = 1$;

(iv$_3$) $\sigma_{e,x}(a) = 0$ if $x \neq e$; (iv$_4$) $\sigma_{e,e}(a) = a$.

In [2] a sort of "skew" or "twisted" monoid ring associated with A and G was constructed by means of the mappings $\sigma_{x,y}$. Examples include group rings, skew polynomial rings, the Weyl algebras and other related ones. There are also connections with gradings of rings [3].

One way of getting a D-structure is from a monoid homomorphism $G \to End(A)$: we define

$$\sigma_{x,y} = \begin{cases} \sigma(x) & \text{if} \quad x = y, \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

There is also a converse.

Theorem 1. *For a monoid G and a unital ring A, a D-structure has all $\sigma_{x,y}$ for $x \neq y$ equal to the zero map if and only if there is a homomorphism $\sigma : G \to End(A)$ with $\sigma_{x,x} = \sigma(x)$ for all $x \in G$.*

The modified monoid ring $A < G; \sigma >$ in this case has the multiplication

$$(a \cdot x)(b \cdot y) = (a\sigma(x)(b)) \cdot xy$$

for $a, b \in A$, $x, y \in G$, rather than $(a \cdot x)(b \cdot y) = ab \cdot xy$ as in the usual monoid ring $A[G]$.

Proposition 1. *If G' is another monoid, $\sigma' : G' \to End(A)$ is a monoid homomorphism and $\varphi : G \to G'$ is a monoid homomorphism, then there is a unique ring homomorphism*

$$\psi : A < G; \sigma > \longrightarrow A < G'; \sigma' >$$

such that $\psi(ax) = a\varphi(x)$ for all $a \in A$, $x \in G$.

Thus in a suitable sense the correspondence $(G; \sigma) \to A < G'; \sigma' >$ is functorial.

For any endomorphism f of A there is a homomorphism from the free monoid $< x >$ on a single generator to $End(A)$ given by $x^n \mapsto f^n$. The associated monoid ring in this case is a skew polynomial ring of some kind.

Example 1. Let G be the infinite cyclic monoid

$$\left\{x^0 \left(= e\right), x^1, x^2, ..., x^n, ...\right\},$$

R a ring with identity, $R\,[t]$ the usual polynomial ring.

We define $\sigma : G \to End R[t]$ by $\sigma(x^n)(p(t)) = p\left(t^{2^n}\right)$. Then $\sigma(x^n)$ as defined is indeed a ring endomorphism, and σ is a monoid homomorphism. Let

$$\sigma_{mn} = \sigma_{x^m,x^n} = \begin{cases} \sigma(x^n) & if \quad m = n, \\ 0 & if \quad m \neq n. \end{cases}$$

In $R[t]\langle G;\sigma\rangle$ we have $xt = x1 \cdot tx^0 = 1\sigma_{11}(t)\, xx^0 = t^2x$.

Thus we get Example 2.5, [3] by a simpler construction.

Example 2. Similarly if K is a field of prime characteristic p, and for our endomorphism we take the one for which $a \mapsto a^p$ for all $a \in K$, then $K < G;\sigma >$ is the Frobenius polynomial ring in x over K in which $xa = a^p x$ for all $a \in K$.

In these examples we have D-structures essentially defined by individual endomorphisms. There is another way to get D-structures from endomorphisms. In [2] it was shown that if f is a homomorphism, δ an $(f, id)-$ derivation of A, i.e. $\delta(ab) = \delta(a)b + f(a)\delta(b)$, and $\delta \circ f = f \circ \delta$, then we get a D-structure using the free monoid on x and defining $\sigma_{x^m x^n} = \binom{n}{m}\delta^{n-m} \circ f^m$ for $n \geq m$ and all others to be zero. (If $\delta \circ f \neq f \circ \delta$ there is a more complicated D-structure.)

Proposition 2. *Let $f : A \to A$ be an endomorphism, and let $\delta(a) = a - f(a)$ for all $a \in A$. Then δ is an (f, id) and an (id, f) derivation and $\delta \circ f = f \circ \delta$.*

As above we get a D-structure from f and δ and hence, in effect, from f. As a simple illustration we have

Example 3. In $\mathbb{C}$, if $f(x + yi) = x - yi$, then $\delta(x + yi) = 2yi$. Let us note three things about this elementary example.

(1) $f^2 = id$;

(2) $\frac{1}{2}\delta$ exists and is also an (f, id) and an (id, f) derivation which commutes with f and

(3) $\mathbb{C}$ is graded by $\mathbb{Z}_2$.

More generally we have:

Theorem 2. *The following conditions are equivalent for a ring A.*

(i) A has an automorphism f of order ≤ 2 such that $a - f(a) \in 2A$

for all $a \in A$*.*

(ii) A *has an automorphism* f *of order* ≤ 2 *and an idempotent* (f, id) *and* (id, f) *derivation* δ *such that* $a = f(a) + 2\delta(a)$ *for all* $a \in A$*.*

(iii) A *is* $\mathbb{Z}_2-$ *graded.*

The following special cases have been proved by Yu. A. Bahturin and M. M. Parmenter:

(1) If $2A = 0$, then $f = id$ and $\mathbb{Z}_2-$ gradings correspond to idempotent derivations. [4]

(2) If A is $2-$ torsion free, then $\mathbb{Z}_2-$ gradings correspond to automorphisms f of order ≤ 2 such that $a - f(a) \in 2A$ for all $a \in A$ [1].

Full details of our results will appear elsewhere.

References

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