Proceedings of the Third Conference of Mathematical Society of Moldova
IMCS-50, August 19-23, 2014, Chisinau, Republic of Moldova

Methodology of matrix representation of higher order elasticity constants. Fourth order tensor

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Abstract

The constitutive nonlinear equations of anisotropic materials are examined in reversible deformation area. The constitutive equations of the second order, in which the tensors of elastic constants of forth order listed, are analyzed in detail. The matrix representation of these tensors and analysis of independent constants of elasticity in function of material symmetry and type of atoms interactions is given.

Keywords: tensor, stress, strain, symmetry, constant elasticity

1 Introduction

With superior order tensors (four, six, eight) we meet at studying the relations between stress and strain. At reversible process the governing equations are written in the form

$$t_{ij} = c_{ijnm}d_{nm} + c_{ijnmpq}d_{nm}d_{pq} + c_{ijnmpqkl}d_{nm}d_{pq}d_{kl} + ...,$$

where by t_{ij}, d_{ij} – the stress and strain tensors are denoted respectively, but by c_{ijnm}, c_{ijnmpq}, $c_{ijnmpqkl}$ – tensors of elasticity constants of order four, six and eight. From symmetry of stress, strain tensors and thermodynamic lows, for tensors of elastic constants the following symmetry relations result:

$$c_{ijnm} = c_{jinm} = c_{ijmn} = c_{nmij} \tag{1}$$

$$c_{ijnmpq} = c_{jinmpq} = c_{ijmnpq} = c_{ijmnqp} = c_{nmijpq} = c_{pqnmij} = c_{ijpqnm}$$

$$c_{ijnmpqkl} = c_{jinmpqkl} = c_{ijmnpqkl} = c_{ijnmqpkl} =$$

$$= c_{ijnmpqlk} = c_{nmijpqkl} = c_{k\ln mpqij}. \tag{2}$$

In function of type of interaction between atoms or molecules, the extra relations can be added to the relationships (1), (2). If the interactions between atoms or molecules are central (the ionic contact), then tensors of elasticity constants of any order are totally symmetric. Remember, that one tensor is totally symmetric if it is symmetric in order with all pairs of indices. In case of forth order tensor, there exists one more relation $c_{ijnm} = c_{injm}$. The material symmetry, which is quantitatively expressed by symmetry plans and symmetry axes of different order, leads to reduction of number of independent constants of elasticity.

2 A matrix representation of a fourth order tensor

The experimental dates for components of tensors of elasticity constants are presented in crystallographic system of coordinates, in which there are sizes only of the independents. Calculation of elasticity constants in arbitrary system is considerably simplified, if the superior order tensors are represented by composite matrix [1]. So, the fourth order tensor can be presented under the shape $c_{ijnm} = (c_{ij})_{nm}$, where, $(c_{ij})_{nm}$ – is square composite matrix of the second order, every element of which represents also a square matrix (3).

For the fourth order tensor, which enjoys the symmetry properties (1), the components of the composite matrix are expressed only by 21 independent constants. The 21 independent constants can be presented as a column matrix 21x1, the elements of which we denote by a_I, where $I = 1, 2, \ldots, 21$.

$$C := \begin{bmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{11} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{12} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{13} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{21} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{22} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{23} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{31} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{32} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{33} \end{bmatrix} \quad (3)$$

So, the tensor of elasticity constants will be expressed in the following way:

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix} & \begin{pmatrix} a_2 & a_7 & a_8 \\ a_7 & a_9 & a_{10} \\ a_8 & a_{18} & a_{11} \end{pmatrix} & \begin{pmatrix} a_3 & a_8 & a_{12} \\ a_8 & a_{13} & a_{14} \\ a_{12} & a_{14} & a_{15} \end{pmatrix} \\ \begin{pmatrix} a_2 & a_7 & a_8 \\ a_7 & a_9 & a_{10} \\ a_8 & a_{18} & a_{11} \end{pmatrix} & \begin{pmatrix} a_4 & a_9 & a_{13} \\ a_9 & a_{16} & a_{17} \\ a_{13} & a_{17} & a_{18} \end{pmatrix} & \begin{pmatrix} a_5 & a_{10} & a_{14} \\ a_{10} & a_{17} & a_{19} \\ a_{14} & a_{19} & a_{20} \end{pmatrix} \\ \begin{pmatrix} a_3 & a_8 & a_{12} \\ a_8 & a_{13} & a_{14} \\ a_{12} & a_{14} & a_{15} \end{pmatrix} & \begin{pmatrix} a_5 & a_{10} & a_{14} \\ a_{10} & a_{17} & a_{19} \\ a_{14} & a_{19} & a_{20} \end{pmatrix} & \begin{pmatrix} a_6 & a_{11} & a_{15} \\ a_{11} & a_{18} & a_{20} \\ a_{15} & a_{20} & a_{21} \end{pmatrix} \end{bmatrix} \quad (4)$$

If tensor is totally symmetric, the following relations can be $a_8 = a_5,\ a_7 = a_4,\ a_{12} = a_6,\ a_{19} = a_{18},\ a_{13} = a_{10},\ a_{14} = a_{11}$. In the case of orthotropic material and for material with cubic symmetry the matrix of elasticity constants is expressed by (5). The relations between stress

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_6 \end{pmatrix} & \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_{16} & 0 \\ 0 & 0 & a_{18} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{19} \\ 0 & a_{19} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{19} \\ 0 & a_{19} & 0 \end{pmatrix} & \begin{pmatrix} a_6 & 0 & 0 \\ 0 & a_{18} & 0 \\ 0 & 0 & a_{21} \end{pmatrix} \end{bmatrix} \quad C := \begin{bmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_4 \end{pmatrix} & \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a_7 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_4 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_7 \\ 0 & a_7 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_7 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_7 \\ 0 & a_7 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \end{bmatrix} \quad (5)$$

and strain in arbitrary system of coordinates in linear approximation is determined by relation

$$d_{in} = \sum_{k=1}^{3} \sum_{q=1}^{3} \left[\sum_{c=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \sum_{j=1}^{3} \left[r_{ij} r_{nm} r_{kl} r_{qc} (C_{jm})_{lc} \right] t_{kq} \right], \qquad (6)$$

where by r_{ij} the matrix of rotation is denoted, in which the base position of this coordinate system is determined from crystallographic system.

3 Conclusions

The possibility of matrix presentation by superior order tensors essentially simplifies the mathematic modelling of nonlinear behavior of anisotropic materials. It has been found, that governing equations of second order in general case of anisotropy are expressed by 77 independent constants of elasticity.

References

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