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Ordering of jobs with three different processing times in the Mxn Bellman-Johnson problem

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Abstract

Bellman-Johnson *Mxn* scheduling problem with monotone (no decreasing, constant or no increasing) jobs of three different processing times is investigated. Three different classes $C_{3.1}$, $C_{3.2}$ and $C_{3.3}$ of such systems are considered. On the basis of earlier results, the solution for optimal ordering of adjacent or nonadjacent jobs in pairs for each of these classes of systems is obtained. In addition, examples of systems for which it is possible to obtain the optimal solution of ordering all n jobs are done, too.

1 Introduction

Bellman-Johnson *Mxn* scheduling problem in sequential systems [1] – one of the main problem in theory of scheduling [2, 3], is not solved, yet. Solutions for some particular cases only are obtained [1-5, 7, 8] and algorithms for quasi optimal solving of the general problem are proposed [5, 6]. The notion of *monotone jobs* is defined in [7]. There, some results referring to partial or total ordering of no decreasing, constant or no increasing jobs are obtained, too. In article [8], the case of monotone jobs with no more than two different processing times is investigated.

In this paper, some particular cases of partial or total ordering of jobs with no more than three different processing times are investigated. For each such a job, the processing time on first sequence of processing units (servers) is the same, on the second sequence of servers is the same too, although possible different from the first one, and on the third sequence of servers is also the same, although possible different from the first two ones.

2 Preliminary considerations

The Mxn Bellman-Johnson problem foresees the execution of n jobs by a system of M consecutive servers. Each server processes, at any moment of time, only one job and may begin the execution of next job immediately after completion of the current one. Jobs' processing order must be the same on all systems' servers. It is required to determine the order, which assure the minimal total processing time T of the n jobs:

$$T = \max_{1 \le u_1 \le u_2 \le \ldots \le u_{m-1} \le n} \left(\sum_{j=1}^{M} \sum_{k=u_{j-1}}^{u_j} \tau_{ji_k} \right) \to \min, \qquad (1)$$

where τ_{ji_k} is the processing time on server j of job i_k, placed in the schedule on place k, and, also, $u_0 = 1$, $u_M = n$.

Let $\Omega = \{1, 2, 3, ..., n\}$ be the set of all jobs to be processed in the system. From earlier known results referring to jobs ordering, below we address, in particular, to Statement 5 and Consequence 4 from paper [5] and to Statements 2, 3, to Consequence 1, to Statements 4, 8, 5 and 9 from paper [7], which in this paper are described as Statements 1-9, respectively, but without their proof.

Statement 1 [5]. Let, for a pair of jobs α and β from the n ones, the following relations take place

$$\min(\tau_{j\,\alpha}; \tau_{j+1,\,\beta}) \le \min(\tau_{j+1,\,\alpha}; \tau_{j\beta}), j = \overline{1, M-1} \qquad (2)$$

and, at the same time, let for a server $v \in [2, M]$ the inequality

$$\tau_{v\alpha} < \tau_{v\beta} \qquad (3)$$

takes place and for a server $k \in [2, v-1]$ the equality

$$\tau_{k\alpha} = \tau_{k\beta} \qquad (4)$$

takes place; in these conditions, if the inequality

$$\tau_{k\alpha} \geq \tau_{k-1,\alpha} \tag{5}$$

takes place too, then when placing jobs α and β near each other in the schedule it is opportune, in sense of (1), that $\alpha \rightarrow \beta$ (job α precedes to job β).

Statement 2 [5]. If, for any pair α, β from the n jobs, relations (2)-(5) take place and these are transitive ones, then the optimal, in sense of (1), schedule can be obtained according to the rule: $\alpha \rightarrow \beta$, if conditions (2)-(5) are satisfied.

Statement 3 [1, 7]. Conditions (2) are transitive ones, in other words, if relations (2) and relations $\min(\tau_{j\beta}; \tau_{j+1,\gamma}) \leq \min(\tau_{j+1,\beta}; \tau_{j\gamma})$, $j = \overline{1, M-1}$ take place, then relations $\min(\tau_{j\alpha}; \tau_{j+1,\gamma}) \leq \min(\tau_{j+1,\alpha}; \tau_{j\gamma})$, $j = \overline{1, M-1}$ take place, too.

Statement 4 [7]. At $\alpha \in A$, the conditions (3)-(5) are satisfied.

Statement 5 [7]. At $\alpha \in A$, the conditions (2)-(5) are transitive ones.

Statement 6 [7]. If relations

$$\tau_{j\alpha} = \tau_{j+1,\alpha}, \tau_{j\beta} = \tau_{j+1,\beta}, j = \overline{s, u} \tag{6}$$

take place, then the subset of conditions (2) for jobs α and β on the server fragment $[s; u]$ is satisfied.

Statement 7 [7]. When placing jobs $\alpha \in A$ and $\beta \in E$ near each other in the schedule, it is opportune, in sense of (1), that $\alpha \rightarrow \beta$.

Statement 8 [7]. Let $L = \overline{i_l, i_{l+r-1}}$ and $L \subseteq C$, then it is unimportant, in sense of (1), the reciprocal placement of subset's L jobs in the schedule on places $\overline{l, l+r-1}$ – this can be an arbitrary one.

Statement 9 [7]. Let $L = \overline{i_l, i_{l+r-1}}$ and $L \subseteq (A \cup E) \subseteq \Omega$, then the rearrangement in the schedule of different categories of subsets of jobs from L on places $\overline{l, l+r-1}$ is opportune, in sense of (1), according to the order: 1)$L \cap (A \backslash C) \rightarrow L \cap C \rightarrow L \cap (E \backslash C)$ or 2) $L \cap A \rightarrow L \cap (E \backslash C)$ or 3) $L \cap (A \backslash C) \rightarrow L \cap E$.

Below, the following definitions referring to monotone jobs, proposed in paper [7], and the definition regarding monotone jobs with three different processing times are used:

1. *No decreasing jobs* are those from the n ones, for which relations

$$\tau_{ji} \leq \tau_{j+1,i}, \;\; i \in A, \;\; j = \overline{1, M-1} \tag{7}$$

take place. Here A is the set of all no decreasing jobs from the n ones.

2. *No increasing jobs* are those from the n ones, for which relations

$$\tau_{ji} \geq \tau_{j+1,i}, \;\; i \in E, \;\; j = \overline{1, M-1}, \tag{8}$$

take place. Here E is the set of all no increasing jobs from the n ones.

3. *Constants jobs* are those from the n ones, for which relations

$$\tau_{ji} = \tau_i, \;\; i \in C, \;\; j = \overline{1, M}, \tag{9}$$

take place. Here $C = C_1$ is the set of all constant jobs from the n ones. From relations (7) – (9), one can see that

$$C = C_1 \subseteq (A \cup E). \tag{10}$$

4. *Monotone with three different processing times jobs* (of type C_3) are those from the n ones, for which relations

$$\tau_{ji} = \begin{cases} \tau_{1i}, j = \overline{1, \nu_i} \\ \theta_i, j = \overline{v_i + 1, k_i} \\ \tau_{Mi}, j = \overline{k_i + 1, M} \end{cases}, i \in C_3, \tag{11}$$

take place. Here C_3 is the set of all monotone jobs with three different processing times from the n ones. The processing time of job i on first sequence of servers, namely $j = \overline{1, \nu_i}$, is τ_{1i}, on second sequence of servers, namely $j = \overline{\nu_i + 1, k_i}$, is θ_i, and on third sequence of servers, namely $j = \overline{k_i + 1, M}$, is τ_{Mi}. From relations (7), (8) and (11), one can easy observe that relation

$$C_3 \subseteq (A \cup E) \tag{12}$$

takes place.

From the multitude of possible particular cases, referring to the set C_3 of jobs, the following three cases are investigated:

1) $C_3 = C_{3.1}$, where the set $C_{3.1}$ is constituted from n jobs of C_3 type for which the equalities $\tau_{1i} = \tau$, $i = \overline{1,n}$ take place;

2) $C_3 = C_{3.2}$, where the set $C_{3.2}$ is constituted from n jobs of C_3 type for which the equalities $\theta_i = \tau$, $i = \overline{1,n}$ take place;

3) $C_3 = C_{3.3}$, where the set $C_{3.3}$ is constituted from n jobs of C_3 type for which the equalities $\tau_{Mi} = \tau$, $i = \overline{1,n}$ take place.

The proof of statements, with regard to jobs of types $C_{3.1}$, $C_{3.2}$ and $C_{3.3}$ ordering formulated below, is done by confirming the satisfaction of conditions (2)-(5) from Statement 1, for jobs placed near each other in the schedule, or of those of Statement 2, for general ordering of jobs in the schedule. According to their description, conditions (3)-(5) are satisfied at that time, when for the definition domain, outlined by relations (3) and (4), the relation (5) takes place; if this definition domain is empty, then it is not needed to satisfy relation (5) and is considered that conditions (3)-(5) are satisfied.

3 Ordering of $C_{3.2}$ type jobs

Let us consider a particular set $C_{3.2}$ of no decreasing (of type A) or no increasing (of type E) n jobs with the following processing times:

$$\tau_{ji} = \begin{cases} \tau_{1i}, j = \overline{1,\nu_i} \\ \tau, j = \overline{\nu_i + 1, \kappa_i} \\ \tau_{Mi}, j = \overline{\kappa_i + 1, M} \end{cases}, i = \overline{1,n}, \tag{13}$$

accepting, to extend the implicated categories of jobs, that there can be $\kappa_i = \nu_i$, too, when job i is with only two different processing times (τ_{1i} and τ_{Mi}). One example of two jobs α and β of type $C_{3.2}$ is shown in Figure 1.

Statement 10. For the set of jobs defined by relations (13), it is opportune, in sense of (1), that $\alpha \to \beta$ if $\alpha \in A$ and $\beta \in E$ or if relations:

$$\min(\tau_{1\alpha}; \tau_{M\beta}) \leq \min(\tau_{M\alpha}; \tau_{1\beta}), \tag{14}$$

$$\nu_\alpha \geq \nu_\beta, \tag{15}$$

$$\kappa_\alpha \quad \geq \quad \kappa_\beta \tag{16}$$

take place.

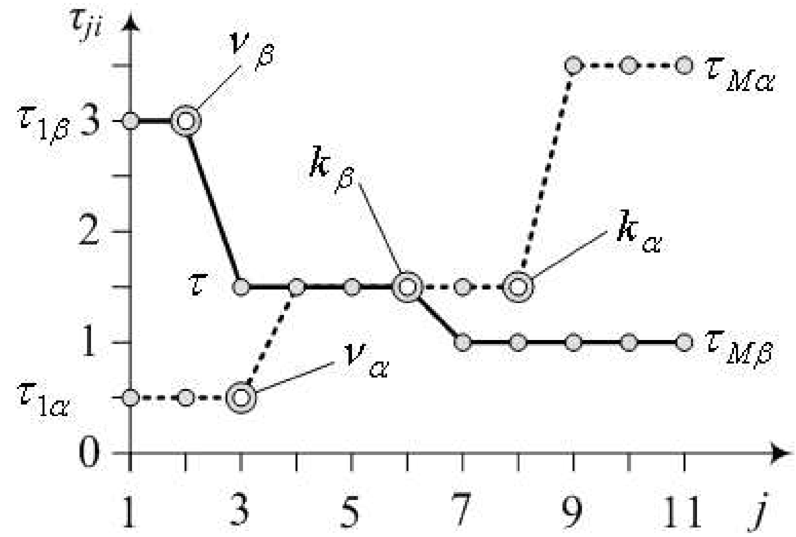

Figure 1. Two jobs α and β of type $C_{3.2}$.

Proof. According to (12), the jobs from the set $C_3 = C_{3.2}$ are monotone no decreasing (belong to set A) or no increasing (belong to set E) ones. At the same time, on the basis of Statement 7, if $\alpha \in A$, $\beta \in E$ and $C_{3.2} = (A \cup E)$, then the placement of jobs α and β in the schedule is opportune, in sense of (1), in order $\alpha \to \beta$. Hence, it remains to prove the reliability of this statement for cases $(\alpha, \beta) \in A$ and $(\alpha, \beta) \in E$. Proof will be done by confirming the satisfaction of conditions (2)-(5) from the Statement 1 and of their transitivity (according to Statement 2).

From relations (13), it is easy to see that $\nu_\alpha \leq \kappa_\alpha$ and $\nu_\beta \leq \kappa_\beta$. Therefore, for two concrete compared jobs α and β, there can be the following six variants of relations among values ν_α, κ_α, ν_β and κ_β:

$$\left.\begin{array}{ll} 1)\ \nu_\alpha \le \kappa_\alpha \le \nu_\beta \le \kappa_\beta; & 2)\ \nu_\alpha \le \nu_\beta \le \kappa_\alpha \le \kappa_\beta; \\ 3)\ \nu_\alpha \le \nu_\beta \le \kappa_\beta \le \kappa_\alpha; & 4)\ \nu_\beta \le \nu_\alpha \le \kappa_\alpha \le \kappa_\beta; \\ 5)\ \nu_\beta \le \nu_\alpha \le \kappa_\beta \le \kappa_\alpha; & 6)\ \nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha. \end{array}\right\} \quad (17)$$

The procedure for confirming the satisfaction of conditions (2) for each of the six variants (17) is the following. Let us consider the variant 1: $\nu_\alpha \le \kappa_\alpha \le \nu_\beta \le \kappa_\beta$. According to Statement 6, the conditions (2) on server fragments $[s; u]$, for which relations (6) take place, are satisfied. That's why, for variant 1 from (17), it is also necessary to verify the following 10 cases:

$$\left.\begin{array}{ll} 1.1)\ j = \nu_\alpha < \kappa_\alpha; & 1.2)\ j = \nu_\alpha = \kappa_\alpha < \nu_\beta; \\ 1.3)\ j = \nu_\alpha = \kappa_\alpha = \nu_\beta < \kappa_\beta; & 1.4)\ j = \nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa_\beta; \\ 1.5)\ \nu_\alpha < j = \kappa_\alpha < \nu_\beta; & 1.6)\ \nu_\alpha < j = \kappa_\alpha = \nu_\beta < \kappa_\beta; \\ 1.7)\ \nu_\alpha < j = \kappa_\alpha = \nu_\beta = \kappa_\beta; & 1.8)\ \kappa_\alpha < j = \nu_\beta < \kappa_\beta; \\ 1.9)\ \kappa_\alpha < j = \nu_\beta = \kappa_\beta; & 1.10)\ \nu_\beta < j = \kappa_\beta. \end{array}\right\} \quad (18)$$

For each of the ten cases from (18), it is needed to verify the respective condition from (2), taking into account relations (13) and (14). For example, for cases 1.5 and 1.7 from (18), one has, respectively:

$$1.5)\ \min(\tau; \tau_{1\beta}) \le \min(\tau_{M\alpha}; \tau_{1\beta}); \quad (19)$$

$$1.7)\ \min(\tau; \tau_{M\beta}) \le \min(\tau_{M\alpha}; \tau_{1\beta}). \quad (20)$$

If $(\alpha, \beta) \in A$ then, on the basis of relations (7), (8) and (13), inequalities $\tau_{1\beta} \le \tau \le \tau_{M\alpha}$ take place, hence condition (19) takes place, too. But the condition (20) doesn't take place, because according to (13) inequalities $\tau \ge \tau_{1\beta}$ and $\tau_{M\beta} \ge \tau_{1\beta}$ take place. In a similar mode, it was established that conditions (19) and (20) for cases 1.1, 1.2, 1.3, 1.4 and 1.5 at $(\alpha, \beta) \in A$ are satisfied.

If $(\alpha, \beta) \in E$ then, according to relations (7), (8) and (13), inequalities $\tau_{M\alpha} \le \tau \le \tau_{1\beta}$ take place, thus condition (19) doesn't take place. At the same time, because of $(\alpha, \beta) \in E$, relation $\tau_{M\beta} \le \tau_{1\beta}$ takes place and, according to relations (14), inequality $\tau_{M\beta} \le \tau_{M\alpha}$ takes place; hence condition (20) takes place, too. In a similar mode, it was established that conditions (19) and (20) for cases 1.4, 1.7, 1.8, 1.9, 1.10 at $(\alpha, \beta) \in E$ are satisfied.

Combining cases $(\alpha, \beta) \in A$ and $(\alpha, \beta) \in E$, it is easy to obtain that conditions (2) for variant 1 take place only at

$$\nu_\alpha = \nu_\beta = \kappa_\alpha = \kappa_\beta, \tag{21}$$

which corresponds to the case with two different processing times for each job: τ_{1i} and τ_{Mi}, $i = \overline{1, n}$.

Obtained results for the six variants (17) are specified in Table 1. In this table, the cases for each of variants 2-6 are formed in a similar mode as the formation of cases for variant 1, but taking into account the particular order, for the concrete variant, of values ν_α, κ_α, ν_β and κ_β.

From Table 1 for each of the six variants, it is easy to observe that even if the set of cases, for which the local conditions are satisfied, is different (depending of job type), however the solution by job types is the same.

Table 1. Cases that satisfy conditions (2) for the set (13) of jobs

Variant	Job type	Cases that satisfy local conditions (2)	Solution by job types	Solution for the variant
1	$(\alpha, \beta) \in A$	1.1-1.5	$\nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa_\beta$	$\nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa_\beta$
	$(\alpha, \beta) \in E$	1.4, 1.7-1.10	$\nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa_\beta$	
2	$(\alpha, \beta) \in A$	2.1-2.4, 2.8, 2.9	$\nu_\alpha = \nu_\beta \le \kappa_\alpha = \kappa_\beta$	$\nu_\alpha = \nu_\beta \le \kappa_\alpha = \kappa_\beta$
	$(\alpha, \beta) \in E$	2.2, 2.4, 2.5, 2.7, 2.9, 2.10	$\nu_\alpha = \nu_\beta \le \kappa_\alpha = \kappa_\beta$	
3	$(\alpha, \beta) \in A$	3.1-3.4, 3.8-3.10	$\nu_\alpha = \nu_\beta \le \kappa_\beta \le \kappa_\alpha$	$\nu_\alpha = \nu_\beta \le \kappa_\beta \le \kappa_\alpha$
	$(\alpha, \beta) \in E$	3.2-3.10	$\nu_\alpha = \nu_\beta \le \kappa_\beta \le \kappa_\alpha$	
4	$(\alpha, \beta) \in A$	4.1-4.9	$\nu_\beta \le \nu_\alpha \le \kappa_\alpha = \kappa_\beta$	$\nu_\beta \le \nu_\alpha \le \kappa_\alpha = \kappa_\beta$
	$(\alpha, \beta) \in E$	4.1, 4.2, 4.4, 4.5, 4.7, 4.9-4.10	$\nu_\beta \le \nu_\alpha \le \kappa_\alpha = \kappa_\beta$	
5	$(\alpha, \beta) \in A$	5.1-5.10	$\nu_\beta \le \nu_\alpha \le \kappa_\beta \le \kappa_\alpha$	$\nu_\beta \le \nu_\alpha \le \kappa_\beta \le \kappa_\alpha$
	$(\alpha, \beta) \in E$	5.1-5.10	$\nu_\beta \le \nu_\alpha \le \kappa_\beta \le \kappa_\alpha$	
6	$(\alpha, \beta) \in A$	6.1-6.10	$\nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha$	$\nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha$
	$(\alpha, \beta) \in E$	6.1-6.10	$\nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha$	

From the last column of Table 1 one can see: the solution for the variant 1 is a particular case of solutions for variants 2-6; the solutions for variants 2-4 are particular cases for the variant 5 solution. Combining the solutions for variants 5 and 6 and taking into account that (see (13)) $\nu_\alpha \leq \kappa_\alpha$ and $\nu_\beta \leq \kappa_\beta$, one can obtain relations (15) and (16).

Thus, from conditions (2)-(5) of Statement 1, it remains to prove that conditions (3)-(5) take place. According to Statement 4, these ones take place at $\alpha \in A$. So, it is needed to prove that conditions (3)-(5) take place at $\alpha \in E$, too, that is at $(\alpha, \beta) \in E$, because case $\{\alpha \in E; \beta \in A\}$ signify that $\beta \to \alpha$ and therefore it secedes. Let $(\alpha, \beta) \in E$ and relations (13)-(16) take place. At $(\alpha, \beta) \in E$, the inequality (3) doesn't take place for server sequences:

- $j = \overline{\kappa_\alpha + 1, M}$, because according to relations (14) the inequality $\tau_{M\alpha} \geq \tau_{M\beta}$ takes place;
- $j = \overline{\kappa_\beta + 1, \kappa_\alpha}$, because according to relations (8) and (13) the inequality $\tau > \tau_{M\beta}$ takes place;
- $j = \overline{\nu_\alpha + 1, \kappa_\beta}$, because according to relations (13) the equalities $\tau_{j\alpha} = \tau_{j\beta} = \tau$ take place;
- $j = \overline{\nu_\beta + 1, \nu_\alpha}$, because according to relations (8) and (13) the relations $\tau_{j\alpha} = \tau_{1\alpha} \geq \tau = \tau_{j\beta}$ take place.

Thus inequality (3) can take place only at $j = \overline{1, \nu_\beta}$, but according to (13) in this case the inequality (4) doesn't take place. So, for $j = \overline{1, M}$, inequalities (3) and (4) don't take place concomitantly and the necessity of satisfaction the condition (5) secedes. Hence conditions (3)-(5), and with them *conditions (2)-(5), too, are satisfied.*

It remains *to prove the transitivity of conditions* (2)-(5). In this aim, it is sufficient to prove the transitivity of conditions (14)-(16), because, as was confirmed above in this section, if conditions (14)-(16) take place, then conditions (2)-(5) take place, too. According to Statement 3, conditions (2) are transitive ones, and conditions (14) are a particular case of conditions (2), so they are transitive, too.

One can easyly observe that conditions (15) and (16) are transitive, too. Really, if relations $\nu_\alpha \geq \nu_\beta$ and $\nu_\beta \geq \nu_\lambda$ take place, then the inequality $\nu_\alpha \geq \nu_\gamma$ takes place, too. In a similar mode, if relations $\kappa_\alpha \geq \kappa_\beta$ and $\kappa_\beta \geq \kappa_\gamma$ take place, then the inequality $k_\alpha \geq k_\gamma$ takes place, too. Hence *conditions (14)-(16) are transitive ones*, that was required to be proved.

Statement 11. Let $L = \overline{i_l, i_{l+r-1}}$, $L \subseteq C_{3.2} \subseteq \Omega$ and for each pair of jobs $(\alpha, \beta) \in L$ the conditions of Statement 10 are satisfied. Then the rearrangement of jobs of subset L in the schedule on places $\overline{l, l+r-1}$ is opportune, in sense of (1), in the following mode: 1) beginning with place l, all jobs of subset $L \cap (A \backslash C)$ are placed in such a way that $\alpha \to \beta$, if $v_\alpha \geq v_\beta$, $\kappa_\alpha \geq \kappa_\beta$ and $\tau_{1\alpha} \leq \tau_{1\beta}$; 2) immediately after jobs of subset $L \cap (A \backslash C)$, the jobs of subset $L \cap C$ are placed in the schedule in an arbitrary mode; 3) immediately after jobs of subset $L \cap C$, the jobs of subset $L \cap (E \backslash C)$ are placed in the schedule in such a way that $\alpha \to \beta$, if $v_\alpha \geq v_\beta$, $\kappa_\alpha \geq \kappa_\beta$ and $\tau_{M\alpha} \geq \tau_{M\beta}$.

Proof. According to (10), relations $C_3 = C_{3.2} \subseteq (A \cup E)$ take place. At the same time, because of $L \subseteq C_{3.2}$, relations $L \subseteq (A \cup E)$ hold, too. The opportunity of ordering the categories of jobs of types A or E from L in order $L \cap (A \backslash C) \to L \cap C \to L \cap (E \backslash C)$ results from Statement 9. Here, unlike conditions from Statement 11, the jobs of category C are separated from subsets of jobs of types A and E. With regard to the order of jobs of the same type $A \backslash C$ or $E \backslash C$, the conditions from Statement 11 coincide with those ones from Statement 10, if for pairs of jobs of type $A \backslash C$ ($(\alpha, \beta) \in A \backslash C$) to substitute the condition (14) by the $\tau_{1\alpha} \leq \tau_{1\beta}$ one and for pairs of jobs of type $E \backslash C$ ($(\alpha, \beta) \in E \backslash C$) to substitute the condition (14) by the $\tau_{M\alpha} \geq \tau_{M\beta}$ one; the relevancy of such substitutions is proved in paper [4]. According to Statement 8, the jobs of category C can be placed in the schedule, on places between jobs of category $A \backslash C$ and those of category $E \backslash C$, in an arbitrary mode, that was required to be proved.

Consequence 1. If $L = C_{3.2} = \Omega$ and for each pair of jobs $(\alpha, \beta) \in L$ the conditions from Statement 10 are satisfied yet, then jobs ordering according to the modality, defined in Statement 11, results with an optimal, in sense of (1), schedule of all jobs from Ω.

The relevancy of Consequence 1 results directly from Statement 11, taking into account that in this case $l = 1$ and $r = n$.

Evidently, the conditions from Statement 10 are not always satisfied for all jobs of type $C_{3.2}$. Two examples, for which the conditions from Statement 10 are satisfied for all jobs of type $C_{3.2}$, are described in Consequences 2 and 3.

Consequence 2. For the set of jobs, defined by relations (13), and in addition

$$\nu_i = \nu,\ \kappa_i = \kappa,\ i = \overline{1,n}, \tag{22}$$

the optimal, in sense of (1), ordering of all n jobs is possible.

Proof. It is easy to observe that conditions (22), although correspond to the conditions (15) and (16) from Statement 10, are symmetric for each pair of jobs from the n ones. So, the optimal, in sense of (1), ordering of the n jobs depends on conditions (14) only. At the same time, conditions (14) can be substituted, according to [4], with: $\alpha \to \beta$ if $\tau_{1\alpha} \leq \tau_{1\beta}$ – for pairs of jobs of type A, and $\alpha \to \beta$ if $\tau_{M\alpha} \geq \tau_{M\beta}$ – for pairs of jobs of type E. In that way, there doesn't appear uncertainty with regard to the ordering of jobs of type A and of type E ones, that was required to be proved.

Consequence 3. If for $i \in A \subseteq C_{3.2}$ relations $\tau_{1i} \leq \tau_{1,i+1}$, $\nu_{1i} \geq \nu_{1,i+1}$, $k_{1i} \geq k_{1,i+1}$ take place and for $i \in E \subseteq C_{3.2}$ relations $\tau_{Mi} \geq \tau_{M,i+1}$, $\nu_{1i} \geq \nu_{1,i+1}$, $k_{1i} \geq k_{1,i+1}$ take place, then the optimal, in sense of (1), schedule of all the n jobs is: $1 \to 2 \to 3 \to ... \to n-1 \to n$.

Proof. It is easy to observe that the conditions from Statement 10 are satisfied for each pair of the n jobs defined in Consequence 3.

4 Ordering of $C_{3.1}$ type jobs

Consider a particular set $C_{3.1}$ of n no decreasing (of type A) or no increasing (of type E) jobs with the following processing times:

$$\tau_{ji} = \begin{cases} \tau, j = \overline{1,\nu_i} \\ \theta_i, j = \overline{\nu_i + 1, \kappa_i} \\ \tau_{Mi}, j = \overline{\kappa_i + 1, M} \end{cases},\ i = \overline{1,n}, \tag{23}$$

accepting, to extend the implicated categories of jobs, that there can be $\kappa_i = \nu_i$, too, when the job i is only with two different processing times (θ_i and τ_{Mi}). One example of two jobs α and β of type $C_{3.1}$ is shown in Figure 2.

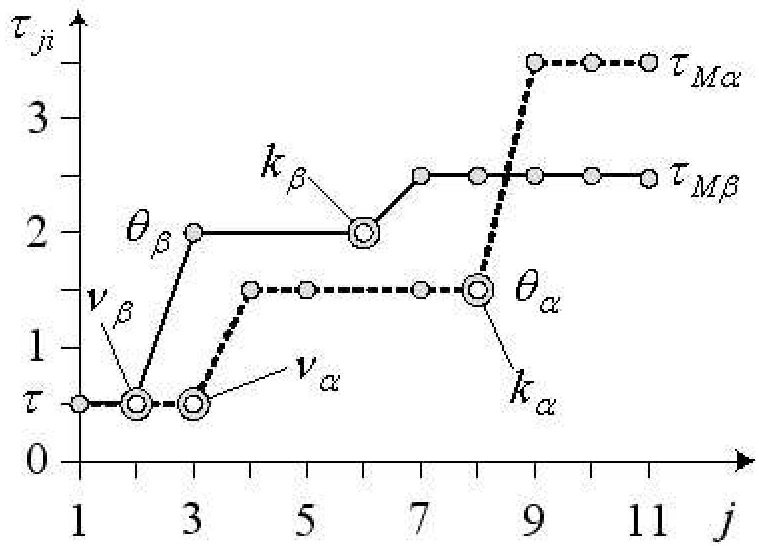

Figure 2. Two jobs α and β of type $C_{3.1}$.

Statement 12. When placing jobs $(\alpha, \beta) \in C_{3.1}$ near each other in the schedule, it is opportune, in sense of (1), that $\alpha \to \beta$ if $\alpha \in A$ and $\beta \in E$ or if there take place the relations

$$\min(\theta_\alpha; \tau_{M\beta}) \leq \min(\tau_{M\alpha}; \theta_\beta) \tag{24}$$

and the conditions of one of the following cases:

a) $\nu_\beta \leq \kappa_\beta \leq \nu_\alpha \leq \kappa_\alpha$; (25)

b) $\nu_\beta \leq \nu_\alpha \leq \kappa_\beta \leq \kappa_\alpha$ and: $(\alpha, \beta) \in A$ or $\{(\alpha, \beta) \in E; \theta_\alpha \geq \theta_\beta\}$; (26)

c) $\nu_\beta \leq \nu_\alpha \leq \kappa_\alpha \leq \kappa_\beta$ and: $\{(\alpha, \beta) \in A; \tau_{M\alpha} \leq \theta_\beta\}$ or $\{(\alpha, \beta) \in E; \tau_{M\alpha} \geq \theta_\beta\}$. (27)

Proof. According to (12), the jobs of $C_3 = C_{3.2}$ set are monotone no decreasing (belong to set A) or no increasing (belong to set E) ones. At the same time, according to Statement 7, if $\alpha \in A$ and $\beta \in E$ when placing jobs α and β near each other in the schedule it is opportune, in sense of (1), that $\alpha \to \beta$. Thus it remains to prove the reliability of the statement for cases $(\alpha, \beta) \in A$ and $(\alpha, \beta) \in E$. The proof will be done by confirming the satisfaction of conditions (2)-(5) from Statement 1.

From (23), one can see that $\nu_\alpha \leq \kappa_\alpha$ and $\nu_\beta \leq \kappa_\beta$. That's why, for two concrete compared jobs α and β, there can be the same six variants (17) of relations among values ν_α, κ_α, ν_β and κ_β as at proving the Statement 10.

The procedure for confirming the satisfaction of conditions (2), for each of the six variants (17), is similar to that used when proving Statement 10, with the difference that, in place of relations (14), the relations (24) will be taken into account. The obtained results for the six variants (17), are described in Table 2. In this table, the cases for each of variants 1-6 are formed in a similar mode as the formation of analog cases when proving the Statement 10.

One can see from Table 2 that, unlike the jobs' set (13), for each of the six variants of the jobs' set, defined by relations (23), there exist many cases when the solutions by jobs type (local ones) differ; at the same time, solutions coincide or are larger for jobs of type A ($(\alpha, \beta) \in A$), than for ones of type E ($(\alpha, \beta) \in E$).

From the last column of Table 1, one can observe that the solutions of variants 1-5 are particular cases of the solution of **variant 6**. So, the general solution coincides with that of variant 6. As well, solutions by jobs type 1, 3, 7, 11, 15 and 19 (see penultimate column of Table 2) coincide with the solution of variants, to which these belong, hence are particular cases of the general solution; local solutions 21 and 22 coincide with the general one. While satisfying some supplementary conditions, there are other cases for which it is possible the partial ordering of jobs, too (see the penultimate column referring to numbered local solutions). It is easy to observe that a part of solutions by jobs types are particular cases of other local solutions or of the general one. The correspondence among these solutions is shown in Table 3.

Table 2. Cases that satisfy conditions (2) for the set $C_{3.2}$ of jobs

Variant	Job type	Cases that satisfy local conditions (2)	Solution by job types	Solution for the variant
1	$(\alpha,\beta)\in A$	1.1-1.5	1. $\nu_\alpha=\kappa_\alpha=\nu_\beta=\kappa_\beta$	$\nu_\alpha=\kappa_\alpha=\nu_\beta=\kappa_\beta$
		1.1-1.5, 1.10 at $\tau_{M\alpha}\le\theta_\beta$	2. $\nu_\alpha=\kappa_\alpha=\nu_\beta\le\kappa_\beta$	
	$(\alpha,\beta)\in E$	1.4, 1.7, 1.9, 1.10	3. $\nu_\alpha=\kappa_\alpha=\nu_\beta=\kappa_\beta$	
		1.3, 1.4, 1.6-1.10 at $\tau_{M\alpha}\ge\theta_\beta$	4. $\nu_\alpha=\kappa_\alpha=\nu_\beta\le\kappa_\beta$	
2	$(\alpha,\beta)\in A$	2.1-2.4, 2.8, 2.9	5. $\nu_\alpha=\nu_\beta\le\kappa_\alpha=\kappa_\beta$	$\nu_\alpha=\nu_\beta=\kappa_\alpha=\kappa_\beta$
		2.1-2.4, 2.8, 2.9, 2.10 at $\tau_{M\alpha}\le\theta_\beta$	6. $\nu_\alpha=\nu_\beta\le\kappa_\alpha\le\kappa_\beta$	
	$(\alpha,\beta)\in E$	2.4, 2.7, 2.9, 2.10	7. $\nu_\alpha=\nu_\beta=\kappa_\alpha=\kappa_\beta$	
		2.2, 2.4, 2.5, 2.7, 2.9, 2.10 la $\theta_\alpha\ge\theta_\beta$	8. $\nu_\alpha=\nu_\beta\le\kappa_\alpha=\kappa_\beta$	
		2.2-2.10 at $\tau_{M\alpha}\ge\theta_\beta$	9. $\nu_\alpha=\nu_\beta\le\kappa_\alpha\le\kappa_\beta$	
3	$(\alpha,\beta)\in A$	3.1-3.4, 3.8-3.10	10. $\nu_\alpha=\nu_\beta\le\kappa_\beta\le\kappa_\alpha$	$\nu_\alpha=\nu_\beta=\kappa_\beta\le\kappa_\alpha$
	$(\alpha,\beta)\in E$	3.3, 3,4, 3.6-3.10	11. $\nu_\alpha=\nu_\beta=\kappa_\beta\le\kappa_\alpha$	
		3.2-3.10 at $\theta_\alpha\ge\theta_\beta$	12. $\nu_\alpha=\nu_\beta\le\kappa_\beta\le\kappa_\alpha$	
4	$(\alpha,\beta)\in A$	4.1-4.9	13. $\nu_\beta\le\nu_\alpha\le\kappa_\alpha=\kappa_\beta$	$\nu_\beta\le\nu_\alpha=\kappa_\alpha=\kappa_\beta$
		4.1-4.10 at $\tau_{M\alpha}\le\theta_\beta$	**14.** $\nu_\beta\le\nu_\alpha\le\kappa_\alpha\le\kappa_\beta$	
	$(\alpha,\beta)\in E$	4.1, 4.4, 4.7, 4.9-4.10	15. $\nu_\beta\le\nu_\alpha=\kappa_\alpha=\kappa_\beta$	
		4.1, 4.2, 4.4, 4.5, 4.7, 4.9, 4.10 at $\theta_\alpha\ge\theta_\beta$	16. $\nu_\beta\le\nu_\alpha\le\kappa_\alpha=\kappa_\beta$	
		4.1-4.10 at $\tau_{M\alpha}\ge\theta_\beta$	**17.** $\nu_\beta\le\nu_\alpha\le\kappa_\alpha\le\kappa_\beta$	
5	$(\alpha,\beta)\in A$	5.1-5.10	**18.** $\nu_\beta\le\nu_\alpha\le\kappa_\beta\le\kappa_\alpha$	$\nu_\beta\le\nu_\alpha=\kappa_\beta\le\kappa_\alpha$
	$(\alpha,\beta)\in E$	5.1, 5.3, 5.4, 5.6-5.10	19. $\nu_\beta\le\nu_\alpha=\kappa_\beta\le\kappa_\alpha$	
		5.1-5.10 at $\theta_\alpha\ge\theta_\beta$	**20.** $\nu_\beta\le\nu_\alpha\le\kappa_\beta\le\kappa_\alpha$	
6	$(\alpha,\beta)\in A$	6.1-6.10	**21.** $\nu_\beta\le\kappa_\beta\le\nu_\alpha\le\kappa_\alpha$	$\nu_\beta\le\kappa_\beta\le\nu_\alpha\le\kappa_\alpha$
	$(\alpha,\beta)\in E$	6.1-6.10	**22.** $\nu_\beta\le\kappa_\beta\le\nu_\alpha\le\kappa_\alpha$	

From the last column of Table 3, one can observe that there are five different cases, defined by relations among values ν_α, κ_α, ν_β and κ_β, which correspond to the general solution of variant 6 and to local solutions 14, 17, 18 and 20. These solutions correspond to cases (25)-(27), hence *conditions (2) are satisfied*.

It is needed still *to prove the satisfaction of conditions* (3)-(5). Case $\{\alpha\in E;\ \beta\in A\}$, which leads according to Statement 7 to the order $\beta\to\alpha$, secedes. Thus there remain cases that cover local solutions 14, 17, 18, 20, 21 and 22 from Table 2. According to Statement 4, these conditions take place for cases that are applicable at $(\alpha,\beta)\in A$ and, namely, those which cover the solutions 14, 18 and 21 from Table 2.

Table 3. Local solutions-particular cases of generalizing solutions for $C_{3.1}$ set

Applicability domain	Local solutions-particular cases (from Table 2)	Generalizing solutions (from Table 2)
$(\alpha,\beta) \in A$ or $(\alpha,\beta) \in E$	1, 3, 7, 11, 15, 19, 21, 22	Var.6. $\nu_\beta \leq \kappa_\beta \leq \nu_\alpha \leq \kappa_\alpha$
$(\alpha,\beta) \in A$	5, 10, 13	18. $\nu_\beta \leq \nu_\alpha \leq \kappa_\beta \leq \kappa_\alpha$
$(\alpha,\beta) \in E$ and $\theta_\alpha \geq \theta_\beta$	8, 12, 16	20. $\nu_\beta \leq \nu_\alpha \leq \kappa_\beta \leq \kappa_\alpha$
$(\alpha,\beta) \in A$ and $\tau_{M\alpha} \leq \theta_\beta$	2, 6	14. $\nu_\beta \leq \nu_\alpha \leq \kappa_\alpha \leq \kappa_\beta$
$(\alpha,\beta) \in E$ and $\tau_{M\alpha} \geq \theta_\beta$	4, 9	17. $\nu_\beta \leq \nu_\alpha \leq \kappa_\alpha \leq \kappa_\beta$

With regard to the other local cases (17, 20 and 22) from Table 2 applicable at $(\alpha,\beta) \in E$, at first it is needed to select server sequences, for which definition domains outlined by relations (3) and (4) are not empty. At $(\alpha,\beta) \in E$, on the base of relations (24), the inequality $\tau_{M\alpha} \geq \tau_{M\beta}$ takes place and, taking into account relation (23), the condition (3) can take place only in the frame of server sequence $j = \overline{\nu_\beta+1,\kappa_\beta}$. Let the condition (3) be satisfied, then the condition (4) can take place only in the frame of server sequence $j = \overline{1,\nu_\beta}$. Let the conditions (3) and (4) take place, then condition (5) is satisfied, too, because in the frame of server sequence $j = \overline{1,\nu_\beta}$, according to (23), equalities $\tau_{j\alpha} = \tau_{j\beta} = \tau$ take place, that was required to be proved.

Statement 13. For the set of n jobs of type $C_{3.1}$, defined by relations (23), it is opportune, in sense of (1), that $\alpha \to \beta$ if $\alpha \in A$ and $\beta \in E$ or if there take place the relations (24) and conditions (25) at $(\alpha,\beta) \in E$ or conditions

$$\nu_\beta \leq \nu_\alpha,\ \ \kappa_\beta \leq \kappa_\alpha \text{ at } (\alpha,\beta) \in A. \tag{28}$$

Proof. It is easy to see that the conditions from Statement 13 are a subset of conditions from Statement 12. Therefore, conditions from Statement 13 satisfy conditions (2)-(5) from Statement 1. In that way, from the same considerations as when proving the Statement 10, it remains to prove that conditions (24), (25) at $(\alpha,\beta) \in E$ and (28) are transitive.

The transitivity of conditions (24) is confirmed by Statement 3. With regard to conditions (25), let $\alpha \to \beta$ and $\beta \to \gamma$, then from the problem conditions we have: $\nu_\beta \leq \kappa_\beta \leq \nu_\alpha \leq \kappa_\alpha$ and $\nu_\gamma \leq \kappa_\gamma \leq \nu_\beta \leq \kappa_\beta$. Combining these two groups of inequalities, one can obtain $\nu_\gamma \leq \kappa_\gamma \leq \nu_\beta \leq \kappa_\beta \leq \nu_\alpha \leq \kappa_\alpha$, from where, eliminating factors ν_β and κ_β referring to job β, results that inequalities $\nu_\gamma \leq \kappa_\gamma \leq \nu_\alpha \leq \kappa_\alpha$ take place, hence conditions (25) are transitive.

It remains to prove the transitivity of conditions (28). Let $(\alpha, \beta, \gamma) \in A$ and $\alpha \to \beta$, $\beta \to \gamma$, then from the problem conditions we have: $\nu_\beta \leq \nu_\alpha$, $\kappa_\beta \leq \kappa_\alpha$ and $\nu_\gamma \leq \nu_\beta$, $\kappa_\gamma \leq \kappa_\beta$. Combining in respective way these two pairs of inequalities, it is easy to obtain $\nu_\gamma \leq \nu_\beta \leq \nu_\alpha$ and $\kappa_\gamma \leq \kappa_\beta \leq \kappa_\alpha$, from where, eliminating factors ν_β and κ_β referring to job β, results that inequalities $\nu_\gamma \leq \nu_\alpha$ and $\kappa_\gamma \leq \kappa_\alpha$ take place, hence conditions (28) are transitive, that was required to be proved.

Consequence 4. If relations $\tau_{1i} \leq \tau_{1,i+1}$, $\nu_{1i} \geq \nu_{1,i+1}$, $k_{1i} \geq k_{1,i+1}$ for $(i, i+1) \in A \subseteq C_{3.2}$ and relations $\tau_{Mi} \geq \tau_{M,i+1}$, $\theta_i \geq \theta_{i+1}$, $\nu_{1i} \geq \nu_{1,i+1}$, $k_{1i} \geq k_{1,i+1}$ for $(i, i+1) \in E \subseteq C_{3.2}$ take place, then the optimal, in sense of (1), ordering of all the n jobs is: $i \in A \to j \in E$, $1 \to 2 \to 3 \to ... \to n-1 \to n$.

Proof. It is easy to observe that the conditions from Statement 12 are satisfied for any pair of the n jobs defined by Consequence 3. The transitivity of conditions, defined in Consequence 4, can be confirmed in a similar mode as the conditions from Statement 13, that was required to be proved.

5 Ordering of $C_{3.3}$ type jobs

Let us consider a particular set $C_{3.3}$ of n no decreasing (of type A) or no increasing (of type E) jobs with following processing times:

$$\tau_{ji} = \begin{cases} \tau_{1i}, j = \overline{1, \nu_i} \\ \theta_i, j = \overline{\nu_i + 1, \kappa_i} \\ \tau, j = \overline{\kappa_i + 1, M} \end{cases}, i = \overline{1, n}, \qquad (29)$$

accepting, to extend implicated categories of jobs, that there can be $\kappa_i = \nu_i$, too, when job i is only of two different processing times (τ_{1i} and θ_i). One example of two jobs α and β of type $C_{3.3}$ is shown in Figure 3.

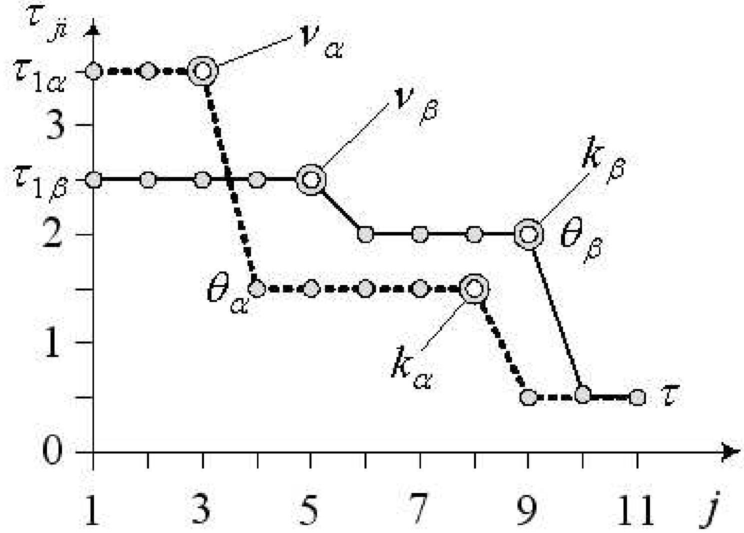

Figure 3. Two jobs α and β of type $C_{3.3}$.

Statement 14. When placing jobs $(\alpha, \beta) \in C_{3.3}$ near each other in the schedule, it is opportune, in sense of (1), that $\alpha \to \beta$ if $\alpha \in A$ and $\beta \in E$ or if there take place the relations

$$\min(\tau_{1\alpha}; \theta_\beta) \leq \min(\theta_\alpha; \tau_{1\beta}) \tag{30}$$

and the conditions of one of the following cases:

a) $\nu_\beta \leq \kappa_\beta \leq \nu_\alpha \leq \kappa_\alpha$; (31)

b) $\nu_\beta \leq \nu_\alpha \leq \kappa_\beta \leq \kappa_\alpha$ and: $(\alpha, \beta) \in E$ or $\{(\alpha, \beta) \in A; \theta_\alpha \leq \theta_\beta\}$; (32)

c) $\nu_\alpha \leq \nu_\beta \leq \kappa_\beta \leq \kappa_\alpha$ and: $\{(\alpha, \beta) \in A; \theta_\alpha \leq \tau_{1\beta}\}$ or $\{(\alpha, \beta) \in E; \theta_\alpha \geq \tau_{1\beta}\}$. (33)

Proof. According to (12), the jobs of set $C_3 = C_{3.3}$ are monotone no decreasing (of type A) or no increasing (of type E). At the same time, according to Statement 7, if $\alpha \in A$ and $\beta \in E$, then when placing jobs α and β near each other in the schedule it is opportune, in sense of (1), that $\alpha \rightarrow \beta$. Thus it remains to prove the reliability of the statement for cases $(\alpha, \beta) \in A$ and $(\alpha, \beta) \in E$. Proof will be done by confirming the satisfaction of conditions (2)-(5) from Statement 1.

From (29), one can see that $\nu_\alpha \leq \kappa_\alpha$ and $\nu_\beta \leq \kappa_\beta$. Therefore, for two concrete compared jobs α and β, there can be the same six variants (17) of relations among values ν_α, κ_α, ν_β and κ_β as at Statement 10.

The procedure for the verification of satisfaction of the conditions (2) for each of the six variants (17) is similar to that used when proving Statement 10 with the difference that, in place of relations (14), the relations (30) are taken into account. The obtained results for the six variants (17) are described in Table 4. In this table, the cases for each of variants 1-6 are formed in a similar mode as the formation of analog cases when proving Statement 10.

From Table 4 it is easy to see that, unlike of jobs set (13), for each of the six variants of jobs set defined by relations (29), there exist many cases when the solution by job types differ; at the same time, this coincide or is larger for jobs of type E ($(\alpha, \beta) \in E$), than for jobs of type A ($(\alpha, \beta) \in A$).

Comparing the last column of Tables 2 and 4, it is easy to observe that the solutions of variants 1-6 for jobs set defined by relations (23) and the ones for jobs set defined by relations (29) coincide. At the same time, solutions by jobs type, specified in the penultimate column of Tables 2 and 4, don't always coincide.

From the last column of Table 4, one can see that the solutions of variants 1-5 are particular cases of the solution of **variant 6**. Thus, the general solution coincide with that of variant 6 one. As well, solutions by jobs types (local ones) 1, 3, 5, 10, 15 and 18 (see penultimate column of Table 4) coincide with solutions for variants, to which these belong, hence are particular cases of the general solution, and local solutions 21 and 22 coincide with the general one. At the same time, when satisfying some supplementary conditions, other cases, for which the

Table 4. Cases that satisfy conditions (2) for the jobs set (29)

Variant	Job type	Cases that satisfy local conditions (2)	Solution by job types	Solution for the variant
1	$(\alpha,\beta) \in A$	1.1-1.5	1. $\nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa_\beta$	$\nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa_\beta$
		1.1-1.7 at $\theta_\alpha \le \tau_{1\beta}$	2. $\nu_\alpha \le \kappa_\alpha = \nu_\beta = \kappa_\beta$	
	$(\alpha,\beta) \in E$	1.4, 1.7-1.10	3. $\nu_\alpha = \kappa_\alpha = \nu_\beta = \kappa\beta$	
		1.1, 1.4, 1.7-1.10 at $\theta_\alpha \ge \tau_{1\beta}$	4. $\nu_\alpha \le \kappa_\alpha = \nu_\beta = \kappa_\beta$	
2	$(\alpha,\beta) \in A$	2.1-2.4, 2.8	5. $\nu_\alpha = \nu_\beta = \kappa_\alpha = \kappa_\beta$	$\nu_\alpha = \nu_\beta = \kappa_\alpha = \kappa_\beta$
		2.1-2.4, 2.8, 2.9 at $\theta_\alpha \le \theta_\beta$	6. $\nu_\alpha = \nu_\beta \le \kappa_\alpha = \kappa_\beta$	
		2.1-2.9 at $\theta_\alpha \le \tau_{1\beta}$	7. $\nu_\alpha \le \nu_\beta \le \kappa_\alpha = \kappa_\beta$	
	$(\alpha,\beta) \in E$	2.2, 2.4, 2.5, 2.7, 2.9, 2.10	8. $\nu_\alpha = \nu_\beta \le \kappa_\alpha = \kappa_\beta$	
		2.1, 2.2, 2.4, 2.5, 2.7, 2.9, 2.10 at $\theta_\alpha \ge \tau_{1\beta}$	9. $\nu_\alpha \le \nu_\beta \le \kappa_\alpha = \kappa_\beta$	
3	$(\alpha,\beta) \in A$	3.1-3.4, 3.10	10. $\nu_\alpha = \nu_\beta = \kappa_\beta \le \kappa_\alpha$	$\nu_\alpha = \nu_\beta = \kappa_\beta \le \kappa_\alpha$
		3.1-3.4, 3.8-3.10 at $\theta_\alpha \le \theta_\beta$	11. $\nu_\alpha = \nu_\beta \le \kappa_\beta \le \kappa_\alpha$	
		3.1-3.10 at $\theta_\alpha \le \tau_{1\beta}$	**12.** $\nu_\alpha \le \nu_\beta \le \kappa_\beta \le \kappa_\alpha$	
	$(\alpha,\beta) \in E$	3.2-3.10	13. $\nu_\alpha = \nu_\beta \le \kappa_\beta \le \kappa_\alpha$	
		3.1-3.10 at $\theta_\alpha \ge \tau_{1\beta}$	**14.** $\nu_\alpha \le \nu_\beta \le \kappa_\beta \le \kappa_\alpha$	
4	$(\alpha,\beta) \in A$	4.1-4.8	15. $\nu_\beta \le \nu_\alpha = \kappa_\alpha = \kappa_\beta$	$\nu_\beta \le \nu_\alpha = \kappa_\alpha = \kappa_\beta$
		4.1-4.9 at $\theta_\alpha \le \theta_\beta$	16. $\nu_\beta \le \nu_\alpha \le \kappa_\alpha = \kappa_\beta$	
	$(\alpha,\beta) \in E$	4.1, 4.2, 4.4, 4.5, 4.7, 4.9,4.10	17. $\nu_\beta \le \nu_\alpha \le \kappa_\alpha = \kappa_\beta$	
5	$(\alpha,\beta) \in A$	5.1-5.7, 5.10	18. $\nu_\beta \le \nu_\alpha = \kappa_\beta \le \kappa_\alpha$	$\nu_\beta \le \nu_\alpha = \kappa_\beta \le \kappa_\alpha$
		5.1-5.10 at $\theta_\alpha \le \theta_\beta$	**19.** $\nu_\beta \le \nu_\alpha \le \kappa_\beta \le \kappa_\alpha$	
	$(\alpha,\beta) \in E$	5.1-5.10	**20.** $\nu_\beta \le \nu_\alpha \le \kappa_\beta \le \kappa_\alpha$	
6	$(\alpha,\beta) \in A$	6.1-6.10	**21.** $\nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha$	$\nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha$
	$(\alpha,\beta) \in E$	6.1-6.10	**22.** $\nu_\beta \le \kappa_\beta \le \nu_\alpha \le \kappa_\alpha$	

partial ordering of jobs is possible, exist, too (see penultimate column referring to numbered solutions by jobs types – local ones). One can easily observe that a part of solutions by jobs types are particular cases of other local solutions or of the general solution. The correspondence among them is shown in Table 5.

Table 5. Local solutions-particular cases of generalizing solutions for $C_{3.3}$ set

Applicability domain	Local solutions-particular cases (from Table 4)	Generalizing solutions (from Table 4)
$(\alpha,\beta) \in A$ or $(\alpha,\beta) \in E$	1, 3, 5, 10, 15, 18, 21, 22	Var.6. $\nu_\beta \leq \kappa_\beta \leq \nu_\alpha \leq \kappa_\alpha$
$(\alpha,\beta) \in E$	8, 13, 17	20. $\nu_\beta \leq \nu_\alpha \leq \kappa_\beta \leq \kappa_\alpha$
$(\alpha,\beta) \in A$ and $\theta_\alpha \leq \theta_\beta$	6, 11, 16	19. $\nu_\beta \leq \nu_\alpha \leq \kappa_\beta \leq \kappa_\alpha$
$(\alpha,\beta) \in E$ and $\theta_\alpha \geq \tau_{1\beta}$	4, 9	14. $\nu_\alpha \leq \nu_\beta \leq \kappa_\beta \leq \kappa_\alpha$
$(\alpha,\beta) \in A$ and $\theta_\alpha \leq \tau_{1\beta}$	2, 7	12. $\nu_\alpha \leq \nu_\beta \leq \kappa_\beta \leq \kappa_\alpha$

Thus, there are five different cases, defined by relations among values ν_α, κ_α, ν_β and κ_β, that correspond to the general solution of variant 6 and to local solutions 12, 14, 19 and 20. These correspond to cases (31)-(33), hence *conditions (2) are satisfied.*

We have still *to prove the satisfaction of conditions* (3)-(5). The case $\{\alpha \in E; \beta \in A\}$, which leads, according to Statement 4, to the order $\beta \to \alpha$, secedes. Thus there remain cases that cover solutions 12, 14, 19, 20, 21 and 22 from Table 4. According to Statement 4, conditions (3)-(5) take place for cases applicable at $(\alpha,\beta) \in A$, namely that which cover solutions 12, 19 and 21 from Table 4.

With regard to the other cases from Table 4 (14, 20 and 22) applicable at $(\alpha,\beta) \in E$, firstly it is needed to select server sequences, for which definition domains, outlined by relations (3) and (4), are not empty. At $(\alpha,\beta) \in E$, on the basis of relations (30), the inequality $\theta_\alpha \geq \theta_\beta$ takes place and, on the basis of conditions (31)-(33), the inequality $\kappa_\beta \leq \kappa_\alpha$ takes place. Thus, taking into account relation (23), the condition (3) can take place only in the frame of servers sequence $j = \overline{1,\nu_\beta}$: (a) at $\nu_\beta > \nu_\alpha$ and $\theta_\alpha < \tau_{1\beta}$ or (b) at $\nu_\beta \leq \nu_\alpha$ and, respectively, $\tau_{1\alpha} < \tau_{1\beta}$. In the first of these two cases, according to data from Table 4, the condition $\theta_\alpha < \tau_{1\beta}$ doesn't hold for local solution 14, but can take place for local solutions 20 and 22. At the same time, according to data from Table 4, for local solutions 20 and 22 the inequality $\nu_\beta \le \nu_\alpha$ takes place, hence case (a) can't take place. Let the case (b) take place and the condition (3) is satisfied; then the condition (4) can't be held, because the condition (3) takes place for the entire servers sequence $j = \overline{1, \nu_\beta}$. Thus, conditions (3) and (4) don't take place concomitantly; hence conditions (3)-(5) are satisfied that was required to be proved.

Statement 15. For the set of n jobs of type $C_{3.3}$, defined by relations (29), it is opportune, in sense of (1), that $\alpha \to \beta$ if $\alpha \in A$ and $\beta \in E$ or if there take place the relations (30) and conditions (31) at $(\alpha, \beta) \in A$ or conditions

$$\nu_\beta \le \nu_\alpha,\ \kappa_\beta \le \kappa_\alpha \text{ at } (\alpha, \beta) \in E. \tag{34}$$

Proof. We can see that the conditions from Statement 15 are a subset of the ones from Statement 14. Therefore, the conditions from Statement 15 satisfy the conditions (2)-(5) from Statement 1. Thus, from the same considerations as when proving the Statement 10, it remains to prove, that conditions (30), (31) at $(\alpha, \beta) \in A$ and (34) are the transitive ones.

The transitivity of relations (30) is confirmed by Statement 3. With regard to conditions (31), these coincide with the (25) ones and the transitivity of the last are proved in Statement 13. Note, that the proof of transitivity of conditions (34) doesn't depend on the class (A or E) to which the jobs α and β belong. At the same time, if not to take into account the class to which the jobs α and β belong, then relations (34) coincide with those from (28) and the transitivity of last ones is confirmed by Statement 13, that was required to be proved.

Consequence 5. If the relations $\tau_{1i} \le \tau_{1,i+1}$, $\theta_i \le \theta_{i+1}$, $\nu_{1i} \ge \nu_{1,i+1}$, $k_{1i} \ge k_{1,i+1}$ for $(i, i+1) \in A \subseteq C_{3.3}$ and relations $\tau_{Mi} \ge \tau_{M,i+1}$, $\nu_{1i} \ge \nu_{1,i+1}$, $k_{1i} \ge k_{1,i+1}$ for $(i, i+1) \in E \subseteq C_{3.3}$ take place, then the optimal, in sense of (1), ordering of all the n jobs is: $i \in A \to j \in E$, $1 \to 2 \to 3 \to ... \to n-1 \to n$.

Proof. We can easily observe that the conditions from Statement 14 are satisfied for each pair from the n jobs, defined in Consequence 5. At the same time, the transitivity of relations, defined in Consequence 5, can be confirmed in the same mode as of ones from the Statement 15, that was required to be proved.

6 Conclusions

Three classes $C_{3.1}$, $C_{3.2}$ and $C_{3.3}$ of systems with monotone jobs of no more than three different processing times in the *Mxn* Bellman-Johnson ordering problem are investigated. For the class $C_{3.2}$, it is obtained a set of relatively simple rules for partial ordering or, in the case that all jobs satisfy the respective conditions, total ordering of the n jobs. There are obtained the rules for ordering in pairs of adjacent jobs for classes of systems $C_{3.1}$ and $C_{3.3}$, too. Rules for ordering the pairs of jobs, when placing them anywhere in the schedule are defined, too. Examples of concrete systems, for which the optimal order of all n jobs can be obtained, are done, too. The obtained results can be used for jobs ordering in sequential systems, aiming to minimize the total processing time of all jobs.

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