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An abstract approach to the study of derivation mappings on non-commutative rings

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Abstract. An abstract approach to the study of derivation mappings on non-commutative rings is undertaken. These mappings are indexed by elements of multiplicative monoid. We describe completely the derivation mappings in the case of a monoid generated by two elements.

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1 Introduction

Let A be a ring (in general, non-commutative) and $1 \neq 0$, where 0 is the null element of A and 1 is the unit element for multiplication. Let G be a multiplicative monoid in which the unit element is denoted by e.

For each $x \in G$ we consider a family $\sigma_x = (\sigma_{x,y})_{y \in G}$ of mappings $\sigma_{x,y} : A \longrightarrow A$ introduced by the following assumption.

(A) For each $x \in G$ there exists a unique family $\sigma_x = (\sigma_{x,y})_{y \in G}$ of mappings $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,y} = 0$ for almost all $y \in G$ (here and thereafter, almost all will mean all but a finite number, that is, $\sigma_{x,y} \neq 0$ only for a finite set of $y \in G$) and for which the following properties are fulfilled:

(i) $\sigma_{x,y}(a+b) = \sigma_{x,y}(a) + \sigma_{x,y}(b)$ $(a, b \in A; x, y \in G)$;

(ii) $\sigma_{x,y}(ab) = \sum_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$ $(a, b \in A; x, y \in G)$;

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$ $(x, y, z \in G)$;

(iv_1) $\sigma_{x,y}(1) = 0$ $(x \neq y; x, y \in G)$; (iv_2) $\sigma_{x,x}(1) = 1$ $(x \in G)$;

(iv_3) $\sigma_{e,x}(a) = 0$ $(x \neq e; x \in G)$; (iv_4) $\sigma_{e,e}(a) = a$ $(a \in A)$.

The condition (i) implies that $\sigma_{x,y}$ are homomorphisms with respect to addition of the abelian group of the ring A. Due to the properties from (A) the mappings $\sigma_{x,y}$ can be called derivations of the ring A. Thus we say that on the ring A a structure of differentiation σ is defined .

Further, we consider the monoid algebra denoted by $A\langle G\rangle$. Namely, the elements of $A\langle G\rangle$ are mappings $\alpha : G \longrightarrow A$ which we write as

$$\alpha = \sum_{x \in G} a_x \cdot x, \tag{1.1}$$

where $a \cdot x$ denotes the mapping from G into A whose value at x is a and 0 at elements different from x. In (1.1) the sum is taken over almost all $x \in G$, that is the sum is taken over only a finite set. Obviously, $A\langle G\rangle$ is a left A-module if the action of A on $A\langle G\rangle$ is defined by

$$\alpha a = \sum_{x\in G}(aa_x)\cdot x \quad (a \in A).$$

$A\langle G\rangle$ becomes a ring (generally, non-commutative) in respect to the operations

$$\sum_{x\in G} a_x \cdot x + \sum_{x\in G} b_x \cdot x = \sum_{x\in G}(a_x + b_x)\cdot x,$$

$$(a\cdot x)(b\cdot y) = \sum_{z\in G} a\sigma_{x,z}(b)\cdot zy \ (a, b \in A; x, y \in G)$$

and extending with regard to the distributive law by

$$\Big(\sum_{x\in G} a_x \cdot x\Big)\Big(\sum_{x\in G} b_x \cdot x\Big) = \sum_{x\in G}\sum_{y\in G}(a_x\cdot x)(b_y\cdot y).$$

In [1] we construct a category $\mathcal{C}$ in which $A\langle G\rangle$ as an object of it has the universality property. Therefore, it could be applied the well-known construction [2] of skew polynomials of one or several variables over non-commutative ring A.

In the present paper we consider a particular case which is useful in studying skew polynomials of one variable. This particular case refers to the situation in which the monoid G is generated by two elements e and x, where $x \neq e$. This case permits us being in some supplemental assumptions to describe completely mappings of derivation $\sigma_{x,y}$ $(x, y \in G)$ (see Theorem 6 and 8, below). For example, if we consider $\sigma_{n,m} = 0$ for $n < m$, where in convenient notation $\sigma_{nm} := \sigma_{x^n,x^m}$ $(n, m = 0, 1, \dots; x^0 := e)$, then as it turns out that σ_{nn} is a ring-homomorphism for any n, σ_{10} is a $(\sigma_{11}, 1)-$derivation of A (Proposition 2). (The definition of $(\alpha, \beta)-$derivation see, for example, in Cohn [3]). If in addition we assume σ_{11} is an automorphism then the same could be concluded for mappings σ_{nn}. Moreover, the mapping $\delta_0 := \sigma_{21} \circ \sigma_{11}^{-1}$ is a $(\sigma_{11}, 1)-$derivation of A, as well, and generally, the mappings δ_n defined by $\delta_n = \delta_0 \circ \sigma_{nn}$ ($\circ$ denotes the composition of mappings) for any $n = 0, 1, 2, \dots$, are $(\sigma_{n+1\ n+1}, \sigma_{nn})-$ derivations of the ring A (see Theorem 5). In Section 2 we give formulas in which σ_{n0} and σ_{nn} are expressed by σ_{10} and σ_{11}. We also consider here the dual case to the previous one, and namely, the case in which $\sigma_{n,m} = 0$ for $n > m$. In certain supplementary conditions the derivation mapping γ_0 given by the relation $\gamma_0 \circ \sigma_{11} = \sigma_{12}$ is a nilpotent mapping. We note that the results of Section 2 generalize some results obtained for skew polynomials in one variables by Smits in [4, 5] (see also Theorem 8.5, pp. 38–39 [3]).

2 Main results

Throughout the paper we study exclusively the particular case in which the monoid G is generated by two elements e and x, where $x \neq e$. This case is especially important in the study of the ring of skew polynomials in one variable.

Supposing the assumption (A) in what follows, we denote

$$\sigma_{nm} := \sigma_{x^n, x^m} \quad (n, m = 0, 1, \ldots). \tag{2.1}$$

Then the formula from the condition (ii) of (A) can be written as follows

$$\sigma_{nm}(ab) = \sum_{j=0}^{\infty} \sigma_{nj}(a)\sigma_{jm}(b) \quad (a, b \in A;\ n, m = 0, 1, \ldots), \tag{2.2}$$

where as before the sum is taken for almost all $j = 0, 1, \ldots$.

Next we form the matrix

$$\sigma(a) = [\sigma_{nm}(a)]_{n,m=0}^{\infty} \quad (a \in A), \tag{2.3}$$

and we have seen that the formula (2.2) can be written as follows

$$\sigma(ab) = \sigma(a)\sigma(b) \quad (a, b \in A),$$

where the multiplication in the right side is the usual multiplication of matrices. In addition, we note also that

$$\sigma(a + b) = \sigma(a) + \sigma(b) \quad (a, b \in A),$$

that follows at once by the condition (i) of (A). Thus, the following assertion can be formulated.

Proposition 1. *The mapping $\sigma : a \longrightarrow \sigma(a)$ determines a matrix representation of A (in general of infinite degree).*

In our notation the formula from the condition (iii) of (A) is written as follows

$$\sigma_{nm} = \sum_{j_1 + \ldots + j_n = m} \sigma_{1j_1} \circ \cdots \circ \sigma_{1j_n} \quad (n, m = 0, 1, \ldots) \tag{2.4}$$

in which $j_k = 0, 1, 2, \ldots$ for $k = 1, \ldots, n$.

1. (The case $(A^{'})$). In this subsection we discuss the following case.

$(A^{'})$ The mappings σ_{nm} defined by (2.1) are such that $\sigma_{nm} = 0$ for $n < m$. In this case the matrix representation σ given by (2.3) has a lower triangular form, and the formula (2.2) is written as the following

$$\sigma_{nm}(ab) = \sum_{j=m}^{n} \sigma_{nj}(a)\sigma_{jm}(b) \quad (a, b \in A;\ n \geq m,\ n, m = 0, 1, 2, \ldots). \tag{2.5}$$

In particular,

$$\sigma_{n0}(ab) = \sum_{j=0}^{n} \sigma_{nj}(a)\sigma_{j0}(b) \quad (a, b \in A;\ n = 0, 1, \ldots). \tag{2.6}$$

For $n = 1$ the formula (2.6) is written as follows

$$\sigma_{10}(ab) = \sigma_{10}(a)\sigma_{00}(b) + \sigma_{11}(a)\sigma_{10}(b),$$

and since, by the condition (iv_4) of (A), $\sigma_{00}(b) = b$ we have

$$\sigma_{10}(ab) = \sigma_{11}(a)\sigma_{10}(b) + \sigma_{10}(a)b \quad (a, b \in A).$$

It should be mentioned that for any $n = 0, 1, 2, \ldots$ the formula (2.5) implies

$$\sigma_{nn}(ab) = \sigma_{nn}(a)\sigma_{nn}(b) \quad (a, b \in A)$$

and the condition (i) of (A) becomes

$$\sigma_{nn}(a + b) = \sigma_{nn}(a) + \sigma_{nn}(b) \quad (a, b \in A).$$

So, σ_{nn} is a ring-homomorphism. Thus we can formulate the following assertion.

Proposition 2. *Under the assumption $(A^{'})$ the mapping σ_{10} is a $(\sigma_{11}, 1)-$ derivation of A.*

It is clear that, due to the assumption $(A^{'})$, the formula (2.4) can be written as follows

$$\sigma_{nm} = \sum_{j_1+\ldots+j_n=m} \sigma_{1j_1} \circ \cdots \circ \sigma_{1j_n} \quad (n, m = 0, 1, 2, \ldots), \tag{2.7}$$

where $j_k = 0, 1 \quad (k = 1, ..., n)$.

In particular,

$$\sigma_{n0} = \sigma_{10}^{n} \quad (n = 0, 1, 2, \ldots), \tag{2.8}$$

and

$$\sigma_{nn} = \sigma_{11}^{n} \quad (n = 0, 1, 2, \ldots). \tag{2.9}$$

Further properties of the mappings σ_{nm} will be obtained by supposing the following assumption.

(B) The mapping σ_{11} is an automorphism of the ring A.

Then, by formula (2.9), the following assertion follows immediately.

Proposition 3. *The mappings σ_{nn} $(n = 0, 1, \ldots)$ are automorphisms of the ring A.*

Next, we define

$$\delta_0 := \sigma_{21} \circ \sigma_{11}^{-1}, \tag{2.10}$$

where σ_{11}^{-1} denotes the inverse of σ_{11}. We observe that (by using the law of multiplication (2.5) and the relations (2.9))

$$\begin{aligned} \delta_0(ab) = (\sigma_{21} \circ \sigma_{11}^{-1})(ab) = \sigma_{21}(\sigma_{11}^{-1}(ab)) = \sigma_{21}(\sigma_{11}^{-1}(a)\sigma_{11}^{-1}(b)) = \\ = \sigma_{21}(\sigma_{11}^{-1}(a))\sigma_{11}(\sigma_{11}^{-1}(b)) + \sigma_{22}(\sigma_{11}^{-1}(a))\sigma_{21}(\sigma_{11}^{-1}(b)) = \\ = (\sigma_{21} \circ \sigma_{11}^{-1})(a)b + (\sigma_{22} \circ \sigma_{11}^{-1})(a)(\sigma_{21} \circ \sigma_{11}^{-1})(b) = \\ = \delta_0(a)b + \sigma_{11}(a)\delta_0(b) = \sigma_{11}(a)\delta_0(b) + \delta_0(a)b, \end{aligned}$$

i.e.

$$\delta_0(ab) = \sigma_{11}(a)\delta_0(b) + \delta_0(a)b \quad (a, b \in A).$$

Thus, we have

Proposition 4. *Under the assumptions (A') and (B) the mapping δ_0 defined by (2.10) represents a $(\sigma_{11}, 1)$ – derivation of the ring A.*

Let us now consider the following mappings

$$\delta_n := \sigma_{21} \circ \sigma_{n-1\ n-1} \quad (n = 1, 2, \ldots)$$

which in accordance with (2.9) can be represented as follows

$$\delta_n = \sigma_{21} \circ \sigma_{11}^{n-1} \quad (n = 1, 2, \ldots).$$

We observe that

$$\delta_n = \sigma_{21} \circ (\sigma_{11}^{-1} \circ \sigma_{nn}) = (\sigma_{21} \circ \sigma_{11}^{-1}) \circ \sigma_{nn} = \delta_0 \circ \sigma_{nn},$$

so that

$$\delta_n = \delta_0 \circ \sigma_{nn} \quad (n = 0, 1, 2, \ldots).$$

Theorem 5. *Under the hypotheses of Proposition 4 the mapping δ_n for each $n = 0, 1, 2, \ldots$ is a $(\sigma_{n+1\ n+1}, \sigma_{nn})$ – derivation of the ring A.*

Proof. By Propositions 3 and 4 we emphasize the fact that σ_{nn} is an automorphism of the ring A and that the mapping δ_0 is a $(\sigma_{11}, 1)$ – derivation of A.

We have

$$\begin{aligned} \delta_n(ab) = (\delta_0 \circ \sigma_{nn})(ab) = \delta_0(\sigma_{nn}(ab)) = \delta_0(\sigma_{nn}(a)\sigma_{nn}(b)) = \\ = \sigma_{11}(\sigma_{nn}(a))\delta_0(\sigma_{nn}(b)) + \delta_0(\sigma_{nn}(a))\sigma_{nn}(b) = \\ = \sigma_{n+1\ n+1}(a)\delta_n(b) + \delta_n(a)\sigma_{nn}(b), \end{aligned}$$

that is

$$\delta_n(ab) = \sigma_{n+1\ n+1}(a)\delta_n(b) + \delta_n(a)\sigma_{nn}(b) \quad (a, b \in A;\ n = 0, 1, 2, \ldots),$$

and the assertion follows. □

Finally, we remark the following consequence of the formula (2.7).

Theorem 6. *Every mapping* σ_{nm} $(m = 0, 1, \ldots, n;\ n = 0, 1, 2, \ldots)$ *is expressed by the derivation* σ_{10} *and the homomorphism* σ_{11}.

Also from the formula (2.7), as it is easy to see, it can be concluded that the mapping σ_{nm} $(m = 0, 1, ..., n)$ represents the coefficient at t^m of the formal extension of the binomial $(\sigma_{10} + \sigma_{11}t)^n$. Thus, taking into account the relations (2.8) for the derivation of higher order σ_{10}^n we can write

$$\sigma_{10}^n(ab) = \sum_{m=0}^{n} \sigma_{nm}(a)\sigma_{10}^m(b) \quad (a, b \in A;\ n = 0, 1, \ldots). \tag{2.11}$$

For the particular case $\sigma_{11} = 1$ from (2.11) it follows the classical formula of Leibnitz

$$\sigma_{10}^n(ab) = \sum_{m=0}^{n} \binom{n}{m} \sigma_{10}^{n-m}(a)\sigma_{10}^m(b) \quad (a, b \in A;\ n = 0, 1, \ldots).$$

2. (The case (A'')). In this subsection we make some remarks on another situation.

Let us assume

(A'') The mappings σ_{nm} defined by (2.1) are such that $\sigma_{nm} = 0$ for $n > m$.

Some formulae and statements are similar to those from the previous case of (A'). We will restrict ourselves only to their enumeration and, concomitantly, we will give other special properties for considered case.

Thus, in the case of (A''), we have the following properties.

1) $\sigma_{nm} = 0$ for $n > m$ $(n, m = 0, 1, 2, \ldots)$;

2) The matrix representation σ of A has an upper triangular form at each element $a \in A$;

3) The formula of multiplication (2.2) becomes to be of the form

$$\sigma_{nm}(ab) = \sum_{j=n}^{m} \sigma_{nj}(a)\sigma_{jm}(b)\ (a, b \in A;\ n \le m,\ n, m = 0, 1, \ldots). \tag{2.12}$$

4) σ_{nn} for each $n = 0, 1, 2, \ldots$ is a ring-homomorphism;

5) The formula (2.4) implies that

$$\sigma_{nn} = \sum_{j_1+\cdots+j_n=n} \sigma_{1j_1} \circ \cdots \circ \sigma_{1j_n} = \sigma_{11} \circ \cdots \circ \sigma_{11} = \sigma_{11}^n,$$

i.e.

$$\sigma_{nn} = \sigma_{11}^n\ (n = 1, 2, \ldots). \tag{2.13}$$

If, in addition, the assumption (B) is supposed, i.e. the mapping σ_{11} is an automorphism of the ring A, then

6) The mappings σ_{nn} $(n = 0, 1, 2, \dots)$ are also automorphisms of A;

7) The mapping

$$\gamma_0 := \sigma_{12} \circ \sigma_{11}^{-1} \tag{2.14}$$

is a $(1, \sigma_{11})$ - derivation of the ring A;

In general,

8) For each $n = 1, 2, \dots$ the mapping defined by

$$\gamma_n := \sigma_{12} \circ \sigma_{11}^{n-1}$$

is a $(\sigma_{nn}, \sigma_{n+1n+1})$ - derivation of the ring A, that is, the following formula

$$\gamma_n(ab) = \sigma_{nn}(a)\gamma_n(b) + \gamma_n(a)\sigma_{n+1n+1}(b) \ (a, b \in A) \tag{2.15}$$

holds for each $n = 0, 1, 2, \dots$.

Note that, by virtue of the relations (2.14), the mapping γ_n can be also expressed as

$$\gamma_n = \sigma_{12} \circ \sigma_{n-1\ n-1} \quad (n = 1, 2, \dots),$$

and that the formula (2.15) can be deduced via (2.2) by using the fact that σ_{nn} for each $n = 0, 1, 2, \dots$ is a ring-homomorphism. Namely, we have

$$\gamma_n(ab) = \sigma_{12}(\sigma_{n-1\ n-1}(ab)) = \sigma_{12}(\sigma_{n-1n-1}(a)\sigma_{n-1n-1}(b)) =$$

$$= \sigma_{11}(\sigma_{n-1n-1}(a))\sigma_{12}(\sigma_{n-1n-1}(b)) + \sigma_{12}(\sigma_{n-1\ n-1}(a))\sigma_{22}(\sigma_{n-1n-1}(b)),$$

and since $\sigma_{22} = \sigma_{11}^2$, so that $\sigma_{22} \circ \sigma_{n-1n-1} = \sigma_{n+1n+1}$, (2.15) follows at once.

In what follows we require additional conditions on the mappings σ_{nm} $(n, m = 0, 1, \dots)$.

(L) The mappings σ_{1j} $(j = 1, 2, \dots)$ are A - independent from the left, that means that if

$$\sum_{j=1}^{\infty} c_j \sigma_{1j}(a) = 0 \tag{2.16}$$

for all $a \in A$, then $c_j = 0$ $(j = 1, 2, \dots)$.

Note that, due to $\sigma_{1j}(1) = 0$ $(j = 2, 3, \dots)$ and $\sigma_{11}(1) = 1$ (cf. (iv_1) and (iv_2) of (A)), from (2.16) for $a = 1$ it follows that $c_1 = 0$. Therefore, without loss of generality, it can be considered that j is changed beginning with 2 and so on going almost everywhere over the set of integer numbers. Thus, the mappings σ_{1j} $(j = 1, 2, \dots)$ are A-independent from the left if and only if the condition

$$\sum_{j=2}^{\infty} c_j \sigma_{1j}(a) = 0 \quad (\forall a \in A)$$

implies $c_j = 0$ $(j = 2, 3, \dots)$.

Similarly, the notion of A - independent from the right can be introduced for the mappings σ_{1j} $(j = 1, 2, \dots)$. Namely, we say that the mappings σ_{1j} $(j = 1, 2, \dots)$ are A-independent from the right if whenever from the condition

$$\sum_{j=2}^{\infty} \sigma_{1j}(a) c_j = 0 \quad (\forall a \in A)$$

it follows $c_j = 0$ $(j = 2, 3, \dots)$.

An alternative of the assumption (L) is the following one.

(R) The mappings σ_{1j} $(j = 1, 2, \dots)$ are A-independent from the right.

Remark. The assumption of the A-independence (L) (respectively (R)) is satisfied for instance if for the mappings σ_{1j} $(j = 1, 2, \dots)$ there exists a family of the elements $a_j \in A$ $(j = 1, 2, \dots)$ such that the elements $\sigma_{1j}(a_j)$ $(j = 1, 2, \dots)$ are not left (resp. right) zero-divisors in A and $\sigma_{1k}(a_j) = 0$ for $j \neq k$ $(j, k = 1, 2, \dots)$. In the particular case of an integral domain A it is sufficient to require that $\sigma_{1j}(a_j) \neq 0$ $(j = 1, 2, \dots)$ and $\sigma_{1k}(a_j) = 0$ for $j \neq k$ $(j, k = 1, 2, \dots)$.

The next assumption will be useful for our further discussion.

(C) There exists a positive integer r such that $\sigma_{1j} = 0$ for $j > r$.

It turns out that the assumption (C) together with the assumption of the independence (R) implies the following property.

Proposition 7. $\sigma_{kj} = 0$ $(k = 1, \dots, r;\ j = r+1, r+2, \dots)$.

Proof. For $k = 1$ the assertion is true via the assumption (C). For the other values the assertion follows from the independence conditions given by the assumption (R). This can be shown as follows. First we observe that (cf. (2.12))

$$\sigma_{1j}(ab) = \sum_{k=1}^{j} \sigma_{1k}(a) \sigma_{kj}(b) \quad (a, b \in A).$$

Then, since $\sigma_{1j}(ab) = 0$ for $j = r+1, r+2, \dots$, we can write

$$\sum_{k=1}^{j} \sigma_{1k}(a) \sigma_{kj}(b) = 0 \quad (j = r+1, r+2, \dots).$$

For $j = r+1$, we obtain

$$\sum_{k=1}^{r} \sigma_{1k}(a) \sigma_{kr+1}(b) = 0$$

from which, by the assumption (R), we get

$$\sigma_{1r+1}(b) = \ldots = \sigma_{rr+1}(b) = 0 \quad (\forall b \in A),$$

i.e. $\sigma_{kr+1} = 0$ $(k = 1, \ldots, r)$.

The same can be concluded for the other values by similar arguments. □

Further, we note the relations (cf. (iii) of (A))

$$\sigma_{n+ms} = \sum_{k+j=s} \sigma_{nk} \circ \sigma_{mj} = \sum_{k+j=s} \sigma_{mj} \circ \sigma_{nk} \ (s = n+m, n+m+1, \ldots), \tag{2.17}$$

where $k = n, n+1, \ldots.$ and $j = m, m+1, \ldots..$

From Proposition 7 we have $\sigma_{rk} = 0$ for $k = r+1, r+2, \ldots.$ Hence, by applying the relations (2.17) for $n = r$ and $m = j$ $(j = 1, \ldots, r)$, one has

$$\sigma_{rr} \circ \sigma_{1j} = \sigma_{1j} \circ \sigma_{rr} \quad (j = 1, \ldots, r), \tag{2.18}$$

or, taking into account the relations (2.13),

$$\sigma_{11}^{r} \circ \sigma_{1j} = \sigma_{1j} \circ \sigma_{11}^{r} \quad (j = 1, \ldots, r).$$

Note that the relations (2.18) can be deduced also by observing that

$$\sigma_{r+1r+j} = \sum_{k+l=r+j} \sigma_{rk} \circ \sigma_{1l} = \sigma_{rr} \circ \sigma_{1j},$$

$$\sigma_{r+1r+j} = \sum_{k+l=r+j} \sigma_{1k} \circ \sigma_{rl} = \sigma_{1j} \circ \sigma_{rr}.$$

Next we discuss some special properties of the mappings σ_{nm}. Again, in virtue of the condition (iii) of (A), we see

$$\sigma_{rr+j-1} = \sum_{k+l=r+j-1} \sigma_{r-1k} \circ \sigma_{1l} = \sigma_{r-1r-1} \circ \sigma_{1j} + \sigma_{r-1r} \circ \sigma_{1j-1} + \sigma_{r-1r+1} \circ \sigma_{1j-2} + \cdots$$

and since $\sigma_{rr+j-1} = 0$ for $j = 2, 3, \ldots$ and also $\sigma_{r-1r+1} = \sigma_{r-1r+2} = \ldots = 0$, it follows the relations

$$0 = \sigma_{r-1r-1} \circ \sigma_{1j} + \sigma_{r-1r} \circ \sigma_{1j-1} \quad (j = 2, 3, \ldots). \tag{2.19}$$

Similarly

$$\sigma_{rr+k} = \sum_{s+t=r+k} \sigma_{1s} \circ \sigma_{r-1t} = \sigma_{1k+1} \circ \sigma_{r-1r-1} + \sigma_{1k} \circ \sigma_{r-1r},$$

and hence

$$\sigma_{1k+1} \circ \sigma_{r-1r-1} + \sigma_{1k} \circ \sigma_{r-1r} = 0 \quad (k = 1, 2, \ldots) \tag{2.20}$$

In virtue of the relations (2.19) and (2.20), we obtain

$$(\sigma_{1k} \circ \sigma_{r-1r-1}) \circ \sigma_{1j} = \sigma_{1k} \circ (\sigma_{r-1r-1} \circ \sigma_{1j}) = -\sigma_{1k} \circ (\sigma_{r-1r} \circ \sigma_{1j-1}) =$$

$$= -(\sigma_{1k} \circ \sigma_{r-1r}) \circ \sigma_{1j-1} = (\sigma_{1k+1} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1},$$

i.e.

$$(\sigma_{1k} \circ \sigma_{r-1r-1}) \circ \sigma_{1j} = (\sigma_{1k+1} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1} \quad (j = 2, 3, \ldots;\ k = 1, 2, \ldots). \quad (2.21)$$

In particular, for $k = 1$, the formula (2.21) becomes to be as follows

$$\sigma_{rr} \circ \sigma_{1j} = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1} \quad (j = 2, 3, \ldots)$$

and thus, by the commutative relations (2.18), we have

$$\sigma_{1j} \circ \sigma_{rr} = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1} \quad (j = 2, 3, \ldots).$$

Then, we can continue

$$\sigma_{1j} \circ \sigma_{rr}^2 = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ (\sigma_{1j-1} \circ \sigma_{rr}) = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ [(\sigma_{12} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-2}] =$$

$$= (\sigma_{12} \circ \sigma_{r-1\ r-1})^2 \circ \sigma_{1j-2},$$

and, by further iteration, we obtain

$$\sigma_{1j} \circ \sigma_{rr}^{j-1} = (\sigma_{12} \circ \sigma_{r-1\ r-1})^{j-1} \circ \sigma_{11} \quad (j = 2, 3, \ldots). \quad (2.22)$$

Substituting in (2.22) $j = r + 1$ and taking into account that $\sigma_{1r+1} = 0$, we obtain

$$(\sigma_{12} \circ \sigma_{r-1r-1})^r \circ \sigma_{11} = 0. \quad (2.23)$$

Since, by the assumption (B), σ_{11} is an automorphism, from (2.23) it follows

$$(\sigma_{12} \circ \sigma_{r-1r-1})^r = 0.$$

Therefore $\gamma_r = \sigma_{12} \circ \sigma_{r-1r-1}$ is a nilpotent mapping of index r.

In virtue of (2.22), we see that the mappings σ_{1j} $(j = 2, 3, \ldots)$ can be expressed by σ_{11} and σ_{12}, namely

$$\sigma_{1j+1} = \gamma_r^j \circ \sigma_{11}^{-rj+1} \quad (j = 1, 2, \ldots).$$

In addition, we note that the mapping $\gamma_0 = \sigma_{12} \circ \sigma_{11}^{-1}$ commutates with σ_{rr}. In fact, from (2.18), in particular, it follows

$$\sigma_{rr} \circ \sigma_{12} = \sigma_{12} \circ \sigma_{rr},$$

from which, multiplying from the right by σ_{11}^{-1}, we have

$$\sigma_{rr} \circ (\sigma_{12} \circ \sigma_{11}^{-1}) = (\sigma_{12} \circ \sigma_{11}^{-1}) \circ (\sigma_{r+1r+1} \circ \sigma_{11}^{-1})$$

or

$$\sigma_{rr} \circ \gamma_0 = \gamma_0 \circ \sigma_{rr}. \quad (2.24)$$

Further, we change in the formula (2.22) the expression $\sigma_{12} \circ \sigma_{r-1\ r-1}$ by $\gamma_0 \circ \sigma_{rr}$, and we get

$$\sigma_{1j} \circ \sigma_{rr}^{j-1} = (\gamma_0 \circ \sigma_{rr})^{j-1} \circ \sigma_{11}.$$

This relation together with the commutative property (2.24) implies

$$\sigma_{1j} \circ \sigma_{rr}^{j-1} = (\gamma_0^{j-1} \circ \sigma_{11}) \circ \sigma_{rr}^{j-1}.$$

Since σ_{rr} is an automorphism, we conclude

$$\sigma_{1j} = \gamma_0^{j-1} \circ \sigma_{11} \quad (j = 2, 3, \ldots). \tag{2.25}$$

In particular, from (2.25) it follows that

$$\gamma_0^r \circ \sigma_{11} = \sigma_{1r+1} = 0,$$

that is $\gamma_0^r = 0$. Moreover, if $\sigma_{1r} \neq 0$, then $\gamma_0^{r-1} \neq 0$.

We formulate the obtained results as follows.

Theorem 8. *Under the assumptions (A''),(B),(C) and (R) the following assertions hold.*

1) The derivation γ_0 is a nilpotent mapping of index r, that is, $\gamma_0^r = 0$, and $\gamma_0^{r-1} \neq 0$ whenever $\sigma_{1r} \neq 0$;

2) $\gamma_r = \sigma_{12} \circ \sigma_{r-1r-1}$ is also a nilpotent mapping of index r;

3) The mappings σ_{1j} $(j = 2, 3, \ldots)$ are expressed by σ_{11} and σ_{12}, and, moreover, for them the relations (2.25) and the commutation relations (2.18) are held.

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