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A Second Order-Cone Programming Relaxation for Days-Off Scheduling Problem

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***Abstract*— This paper relates the numerical solution of one of the well-known problems related to production systems. Namely, it is about Days-Off Scheduling Problem, which is reformulated in terms of non-convex minimization problems with quadratic constraints. After relaxation of the above problem, a model based on second-order cone programming is obtained.**

Keywords—days-off scheduling; binary quadratic programming; days-off scheduling; style; styling (key words)

I. INTRODUCTION

One of the most frequent and at the same time most difficult organizational problems is the days off planning of the employees of the economic production units provision of services or trade such as IT companies, hospitals, restaurants, stores, public transport, etc. The problem is to minimize the number of working days, satisfying the working conditions and individual preferences of employees.

Such problems were first formulated in terms of integer linear programming by Danzig [1] in 1954. Over the years the problem of planning the working hours of employees in different companies has been addressed by many researchers: in the field of public transport services [2,3], hospital staff scheduling [4,5], scheduling days off [6,7,8,9]. The paper [10] presents a detailed study of the classification of workforce planning problems.

The issues discussed are part of the NP-hard class, being combinatorial optimization problems.

In practical situations the problem is an optimization problem in integers which makes it difficult and time consuming to build a good algorithm.

In this paper, the authors treat the model proposed in [11, 12] to maximize the number of consecutive days off for employees of an economic unit, which is based on the construction of a quadratic programming problem.

The following notation is used in this paper:

$$x^T y = \sum_i x_i y_i \qquad (1)$$

for the inner product of column vectors x and y;
x^T denotes in (1) the transposition of x and $x^T$is a row vector;

$\|x\|_2 = \sqrt{x^T x}$– the Euclidean norm of a vector x;

x_i - denotes the i^{th}component of x;
e_i – the vector with all components equal to zero, except the i^{th} component which is equal to one;
I – the identity matrix.

II. THE DAYS OFF PLANNING PROBLEM

This paper will focus on finding a solution for maximizing the number of consecutive days off of the employees of a specific IT company.

The following notation is used:
p – number of emplyees
Ti – number of days-off for the i employee of the week, in

$$i \in \{1, 2, \dots, p\}$$

n_k – number of necessary employees in the day k,

$$k \in \{1,2, \dots, 7\}$$

the constraints for satisfying daily work requirements, such as a specific task to be assigned to a class of workers that have abilities for it

$$c = (c_1, c_2, \dots, c_7)$$

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c_k is the class of workers q_k that is necessary for accomplishing a specific task for the day k

$$q_k \in \{1, 2, \dots, n_k\}, \qquad n_k = \{1, \quad 2, \dots, 7\}$$

For each $i \in \{1, 2, \dots, p\}$ worker it is introduced 7 binary variables x_{ik}, $k \in \{1, 2, 3, \dots, 7\}$, such that

$$x \in \begin{cases} 1, \text{if the k day is an off} - \text{day} \\ \quad \text{for the k worker} \\ \quad 0, \text{otherwise} \end{cases} \tag{2}$$

The matrix

$$X = \begin{pmatrix} x_{11} x_{12} x_{13} x_{14} x_{15} x_{16} x_{17} \\ x_{21} x_{22} x_{23} x_{24} x_{25} x_{26} x_{27} \\ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \ \dots \\ x_{p1} x_{p2} x_{p3} x_{p4} x_{p5} x_{p6} x_{p7} \end{pmatrix} \in R^{px7}$$

The main objective is maximizing the number of consecutive days off in a week.

Then it can be defined the objective function as follows:

$$g(x) = \sum_{i=1}^{p} (\sum_{k=1}^{6} x_{ik}\, x_{ik+1} + x_{i7}\, x_{i7}) =$$

$$= \sum_{i=1}^{p} (\sum_{k=1}^{6} x_{ik}\, x_{ik+1} + x_{i7}\, x_{i7})$$

The column vector is noted

$$x = (x_{11} x_{12} \dots x_{17}, \ \ x_{21} x_{22} \dots x_{27}$$

$$x_{p1} x_{p2} \dots x_{p7}) \in R^n$$

$$n = 7p$$

$$x \in R^n$$

The following notations are used

$$Q = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \in R^{7x7} \tag{3}$$

$$Q = \begin{pmatrix} Q_0 & 0 & \dots & 0 \\ 0 & Q_0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & Q_0 \end{pmatrix} \in R^{nxn} \tag{4}$$

and

$$g(x) = x^T Q x$$

The folowing constraints are defined:

I. For each day k, the number of workers who take this day off is $p - n_k$.

$$\rightarrow \sum_{i=1}^{p} x_{ik} = p - n_k, \qquad k \in \{1, \dots, 7\} \tag{5}$$

The following notations are used

$$\bar{A}_0 = \begin{pmatrix} 1\,0 \cdots 0\,1\,0 \dots 0 \dots\dots 1\,0 \dots 0 \\ 0 \ddots \ddots \vdots\, 0 \ddots \ddots \vdots \dots\dots 0 \ddots \ddots \vdots \\ \vdots \ddots \ddots 0 \vdots \ddots \ddots 0 \dots\dots \vdots \ddots \ddots 0 \\ 0 \dots 0\,0\,0 \dots 0\,1 \dots\dots 0 \dots 0\,1 \end{pmatrix}, \tag{6}$$

$$\bar{b} = \begin{pmatrix} p - n_1 \\ p - n_2 \\ \vdots \\ p - n_7 \end{pmatrix} \tag{7}$$

so

$$\bar{A}_0 x = \bar{b}.$$

It is noted that

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$$\bar{A}_0 = \left(\begin{pmatrix} 1\,0 \ldots 0 \\ 0\,1 \ldots 0 \\ \ldots\ldots\ldots \\ 0\,0 \ldots 1 \end{pmatrix} \begin{pmatrix} 1\,0 \ldots 0 \\ 0\,1 \ldots 0 \\ \ldots\ldots\ldots \\ 0\,0 \ldots 1 \end{pmatrix} \begin{pmatrix} 1\,0 \ldots 0 \\ 0\,1 \ldots 0 \\ \ldots\ldots\ldots \\ 0\,0 \ldots 1 \end{pmatrix} \right)$$

i.e.

$$\bar{A}_0 = (I\ I\ I \ldots I) \in R^{7xn}$$

II. Each worker has T_i days off during the week

$$\left.\begin{matrix} x_{11} + x_{12} + \cdots + x_{17} = T_1 \\ x_{21} + x_{22} + \cdots + x_{27} = T_2 \\ \ldots + \ \ldots \ + \ldots + \ \ldots \ = \ \ldots \\ x_{p1} + x_{p2} + \cdots + x_{p7} = T_p \end{matrix}\right\} \tag{8}$$

with the matrix

$$\bar{A}_1 = \begin{pmatrix} 1 \ldots 1 \ldots 1 0 \ldots 0 \ldots 0 \ldots 0 \ldots 0 \ldots 0 \\ \vdots \quad \vdots \quad \vdots\,\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ 0 \ldots 0 \cdots 0 1 \ldots 1 \cdots 1 \ldots 0 \ldots 0 \ldots 0 \\ \vdots \quad \vdots \quad \vdots\,\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ 0 \ldots 0 \ldots 0 0 \ldots 0 \ldots 0 \ldots 1 \ldots 1 \ldots 1 \end{pmatrix}, \tag{9}$$

and the vector

$$T = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ \vdots \\ T_p \end{pmatrix}$$

Example, for $p = 3$,

$$\left.\begin{matrix} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} = T_1 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} = T_2 \\ x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} + x_{37} = T_3 \end{matrix}\right\}$$

$$\bar{A}_1 = \begin{pmatrix} 1111111 & 0000000 & 0000000 \\ 0000000 & 1111111 & 0000000 \\ 0000000 & 0000000 & 1111111 \end{pmatrix},$$

and

$$A_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow \text{the row } i \\ \\ \\ \end{matrix}$$

$$\bar{A}_1 = (A_1\ A_2\ \ldots\ A_n) \in R^{pxn}$$

$$\bar{A}_1 x = T$$

III. The constraints defined by the tasks of the day is a set:

$$c = (c_1, c_2, \ldots, c_k).$$

Where c_k is the class of workers $q_k \in \{1,2, \ldots, n_k\}$, which is needed to perform the specific task for the day $k \in \{1, 2, \ldots, 7\}$.

These classes are determined by the company and are represented by the matrix $\bar{A}_2 \in \{0,1\}^{2xn}$ such that

$$Q_{ik} = \begin{cases} 1, & \text{the presence of the worker i} \\ & \quad \text{is required on day k} \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

and

$$\bar{A}_2 x = 0$$

Considering the above restrictions and notations, the problem of scheduling consecutive days off of workers is reduced to solving a quadratic programming problem.

III. QUADRATIC PROGRAMMING MODEL [12]

Consider the quadratic programming problem:

$$y(x) = x^T Q x \rightarrow max$$

subject to

$$\begin{cases} \bar{A}_0 x = b, \\ \bar{A}_1 x = T, \\ \bar{A}_2 x = 0, \\ x \in \{0,1\}^n \end{cases} \tag{11}$$

The following notation is used

$$A = \begin{pmatrix} \bar{A}_0 \\ \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} \in R^{mxn},$$

$$m = 14 + p,$$

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$$n = 7p$$

$$b = \begin{pmatrix} b^I \\ T \\ 0 \end{pmatrix} \in R_+^m$$

The problem can be formulated as follows:

$$f(x) = -g(x) = -x^T Q x \to \min$$

subject to

$$Ax = b,$$

$$x \in \{0,1\}^n$$

IV. SOLVING THE PROBLEM OF QUADRATIC PROGRAMMING

Because it their high level of difficuly, to solve these problems obtained by quadratic programming, there are many methods [12, 17,18].

The authors propose to solve these problems using second order - cone programming relaxation.

V. A SECOND ORDER – CONE PROGRAMMING RELAXATION [13]

The conditions that x_i are binary are equivalent to the non-convex quadratic constraints:

$$x_i^2 - x_i = 0,$$

which in turn are equivalent to the following constraints:

$$\left.\begin{array}{c} x_i^2 - x_i \le 0, \\ \sum_i (x_i - x_i^2) \le 0, \end{array}\right\}$$

Define $M_i = e_i e_i^T$ – the matrix whose all entries are zero, except the (i, i) entry which is one:

$$M_i = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \leftarrow i$$

$$\uparrow i$$

The constraint $x_i^2 - x_i \le 0$ can thus be reformulated as:

$$x^T M_i x - e_i^T x \le 0,$$

from which

$$x^T (M_i + I) x - e_i^T x - x^T x \le 0, \qquad (12)$$

where

$$x = (x_1, x_2, \dots, x_n)^T.$$

It easily observed the matrix $M_i + I$is positive definite that it can be rewritten using Cholesky decomposition as follows:

$$M_i + I = L_i^T L_i = L_i^2$$

where

$$L_i = \sum_{s \ne i} e_s e_s^T + \sqrt{2} e_i e_i^T.$$

$$L_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

Letting nonnegative variable $t_0 = x^T x \ge 0$, the relationship (12) can be rewritten as:

$$\omega_i^T \omega_i \le \xi_i \eta_i \qquad (13)$$

where

$$\omega_i = L_i^T y = \sum_{s \ne i} y_s e_s + \sqrt{2} y_i e_i, \ \xi_i = 1$$

and

$$\eta_i = e_i^T y + t_0 \ge 0.$$

A hyperbolic constraint (13) is equivalent to the second-order cone constraints [14]:

$$\left\| \begin{matrix} 1 - e_i^T y - t_0 \\ 2 L_i y \end{matrix} \right\|_2 \le 1 + e_i^T y + t_0. \qquad (14)$$

On the other hand it is observed that the condition

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$$\sum_i (x_i - x_i^2) \leq 0$$

implies the linear constraints:

$$\textstyle\sum_i x_i - t_0 \leq 0\} \quad (15)$$

Finally, it can be mentioned that the quadratic constraint

$$t_0 = y^T y, \quad (16)$$

used above in constructing second-order cones (9) and (10) are non-convex. The constraint (12) is relaxed such that:

$$y^T y \leq t_0.$$

The last constraints are equivalent to following second order cones constraint:

$$\left\| \begin{matrix} 1 - t_0 \\ 2y \end{matrix} \right\|_2 \leq 1 + t_0, \} \quad (17)$$

Thus, the problem (1)-(6) might be relaxed to the following second-order conic programming problem:

$$\sum_{i=1}^{n} x_i \to min$$

subject to

$$Ax = b.$$

and the constraints (14), (15), (17).

VI. CONCLUSIONS

This paper contains the numerical solving of the Days-Off Scheduling Problem reformulated in terms of non-convex minimization problems, to find a solution for days off scheduling problem with the constraints defined by the tasks of the day.

This paper has proposed a solution for scheduling the staff days off by a relaxation of the Days-Off Scheduling Problem in terms of second-order cone programming. This problem can be effectively solved by the interior point algorithm ([15]).

There is specialized software for solving conic optimization problems ([16]). Future work includes implementing the solution for scheduling days off for an IT company staff

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