Journal of Engineering Science — *Vol. XXIX, no. 4 (2022), pp. 38 - 50*
Fascicle — *Industrial Engineering* — *ISSN 2587-3474*
Topic — *Applied Engineering Sciences and Management* — *eISSN 2587-3482*

https://doi.org/10.52326/jes.utm.2022.29(4).03
UDC 519.63:532.516.5

DETERMINATION OF SOME SOLUTIONS OF THE STATIONARY 2D NAVIER-STOKES EQUATIONS

Iurie Baltag *, ORCID: 0000-0003-3426-1287

Technical University of Moldova, 168 Stefan cel Mare Blvd., Chisinau, Republic of Moldova
*Corresponding author: Iurie Baltag, *iurie.baltag@mate.utm.md*

Received: 11. 04. 2022
Accepted: 12. 10. 2022

Abstract. In this paper, various solutions of the stationary Navier-Stokes equations, which describe the planar flow of an incompressible liquid (or gas), are determined, i.e., solutions containing the components of the velocity of flow - the functions *u*, *v* and the created pressure - *P*. The paper contains three proven theorems, as well as various examples and particular examined cases. Applying Theorem 1, we can find various solutions, where the velocity components represent the imaginary and real parts of a differentiable function of a complex variable. Theorem 2 allows us to determine solutions, where the velocity components are expressed by the partial derivatives of the solutions of Laplace's equation of a special form. It is to be mentioned that these theorems give us solutions that do not depend on the viscosity parameter λ. In theorem 3, an original method for obtaining a series of solutions of the Navier-Stokes equations is presented, in which the viscosity coefficient λ participates explicitly; these solutions cannot be obtained by applying Theorems 1 or 2. The paper contains a large number of particular cases examined and examples of exact determined solutions.

Keywords: *stationary two-dimensional Navier-Stokes equations, system of equations with partial derivatives, exact solutions, method of separation of variables, viscosity, pressure, velocity of plane flow of a liquid or gas.*

Rezumat. În această lucrare se determină diverse soluții ale ecuațiilor staționare Navier-Stokes, care descriu curgerea plană a unui lichid (sau gaz) incompresibil, și anume soluții ce conțin componentele vitezei fluxului de curgere - funcțiile *u*, *v* și presiunea creată – *P*. Lucrarea de față conține trei teoreme demonstrate și diverse exemple și cazuri particulare examinate. Aplicând teorema 1, putem afla diverse soluții, în care componentele vitezei reprezintă partea imaginară și cea reală a unei funcții diferențiabile de variabilă complexă. Teorema 2 ne permite să determinăm soluții, în care componentele vitezei sunt exprimate prin derivatele parțiale ale soluțiilor ecuației lui Laplace de o formă specială. Menționăm, că aceste teoreme ne oferă soluții ce nu depind de parametrul vâscozității λ. În teorema 3 este expusă o metodă originală de obținere a unui șir de soluții ale ecuațiilor Navier-Stokes, în care participă în mod explicit coeficientul vâscozității λ; aceste soluții nu pot fi obținute

aplicând teoremele 1 sau 2. Lucrarea conține un număr mare de cazuri particulare examinate și exemple de soluții exacte determinate.

Cuvinte cheie: *ecuații staționare bidimensionale Navier-Stokes, sistem de ecuații cu derivate parțiale, soluții exacte, metoda separării variabilelor, vâscozitate, presiune, viteza fluxului de curgere plană a unui lichid sau gaz.*

1. Introduction

In the present paper, the Navier-Stokes equations are studied in the two-dimensional (2D) case. In this case the Navier-Stokes equations represent a system containing three partial differential equations with three unknown functions.

Until today, the examined problem has not been definitively solved even in the case of stationary equations, that is, equations that describe the processes of the planar flow of a liquid or gas that does not vary in time.

The complexity of the problem lies in the fact that the first two equations in the system are non-linear.

A method is not developed that would allow us to determine all the solutions of this system. Determining the solutions of the system of Navier-Stokes equations is an important mathematical problem and has various applications in fluid and gas mechanics.

The following system of partial differential equations is examined in this paper:

$$\begin{cases} \frac{P_x}{\mu} + uu_x + vu_y = \lambda(u_{xx} + u_{yy}) + F_x \\ \frac{P_y}{\mu} + uv_x + vv_y = \lambda(v_{xx} + v_{yy}) + F_y \\ u_x + v_y = 0 \end{cases} \tag{1}$$

where: $x, y \in R;\ P = P(x, y); F = F(x; y);\ u = u(x, y), v = v(x, y);\ u_x = \frac{\partial u}{\partial x}.$

System (1) describes the stationary processes of planar flow of an incompressible liquid or gas. Regarding the derivation of the equations of system (1) and the meaning of the physical processes described by this system, consult the works [1 - 3].

The unknowns of system (1) are the following three functions: *P*, which represents the created pressure; *u* and *v*, which represent the components of the flow velocity of a liquid or gas.

The given external force is *F* and has a potential nature, that is, its components are equal to the partial derivatives of this force - *Fx* and *Fy*. The constants $\lambda > 0$ and $\mu > 0$ are the parameters determined by the viscosity and density of the examined liquid (gas). We mention here, that the viscosity parameter has the form $\lambda = \frac{c_0}{R_e}, c_0 > 0,$ where R_e is the Reynolds number.

Some exact solutions of the system (1) are obtained in the papers [4] - [7]. In the paper [8] a series of solutions of the examined system are indicated only for the components of the flow velocity, without determining the pressure.

Suppose that in the plane connected domain *D* functions *P*(*x, y*), *u*(*x, y*), *v*(*x, y*) and *F*(*x, y*) admit partial derivatives continuous up to and including the second order, then the theorems 1-3 stated below is just.

2. Solutions, where the velocity components represent the imaginary part and the real part of a differentiable function of complex variable

Theorem. 1. If *f(z)* is a function of complex variable *z = x + iy*, differentiable at any interior point (*x; y*) of domain D, then system (1) admits solutions of the following form in this domain:

$$u = Imf,\ v = Ref;\ P\ =\ [F\ -\ 0{,}5(\ u^2\ \ +\ \ v^2) + C]\mu. \quad (2)$$

where *C* is an arbitrary constant.

Demonstration T. 1. System (1) is equivalent to the following system:

$$\begin{cases} \frac{P_x}{\mu} - F_x + uu_x + vv_x = \lambda\Delta u - v\left(u_y - v_x\right) \\ \frac{P_y}{\mu} - F_y + uu_y + vv_y = \lambda\Delta v + u\left(u_y - v_x\right) \\ u_x + v_y = 0 \end{cases} \quad (3)$$

where $\Delta u = u_{xx} + u_{yy},\ \Delta v = v_{xx} + v_{yy}$.

Noting that $$G = \frac{1}{\mu}\ P - F + 0{,}5(u^2 + v^2) \quad (4)$$

Then from (3) results that $$\begin{cases} G_x = \lambda\Delta u - v(u_y - v_x) \\ G_y = \lambda\Delta v + u\left(u_y - v_x\right) \\ u_x + v_y = 0 \end{cases} \quad (5)$$

Thus, system (1) is equivalent to system (5). Since *Gxy* = *Gyx* we derive the first equation from (5) in relation to *y*, and the second in relation to *x* and equate the right sides of the obtained equations. As a result, we obtain the following equation for determining the functions *u* and *v*:

$$\lambda\Delta(u_y - v_x) - u(u_y - v_x)_x - v(u_y - v_x)_y = 0 \quad (6)$$

Besides this, u and v have to verify also the last equation from the system (5):

$$u_x + v_y = 0. \quad (7)$$

Therefore, the functions *u* and *v* can be determined separately, independently of the pressure *P*, from the system, which consists of equations (6) and (7).

May it be $u = Imf,\ v = Ref,\ \ f = v(x;y) + iu(x;y),$ were *f* is a function of complex variable *z = x + iy*, differentiable at any interior point (*x; y*) of domain D. Then from Cauchy – Riemann conditions [9] we obtain:

$$\begin{cases} v_x = u_y \\ v_y = -u_x \end{cases} \Leftrightarrow \begin{cases} u_y - v_x = 0 \\ u_x + v_y = 0 \end{cases} \quad (8)$$

The second equation in (8) coincides with (7), and from the first it follows that these functions verify equation (6). Since the functions *u* and *v* admit continuous derivatives up to the second order inclusive in D, they have continuous mixed derivatives in this domain and from (8) we deduce that they are solutions of Laplace's equation, meaning $\Delta u = u_{xx} + u_{yy} = 0, \Delta v = 0$.

Then from (5) we obtain that $\begin{cases} G_x = 0 \\ G_y = 0 \end{cases} \Rightarrow G(x;y) = C - \text{const}.$

We substitute this result in (4) and express the pressure *P*. **T. 1.** is proved.

Below we will give 2 examples of determining the solutions of system (1) according to theorem 1.

Example 1. If $f(z) = e^{z^2}$, then we obtain the following solutions of system (1):

$$\begin{cases} u = e^{x^2-y^2} sin(2xy); v = e^{x^2-y^2} cos(2xy), \\ P = (F - 0{,}5e^{2(x^2-y^2)} + C)\mu \end{cases} D = OXY. \tag{9}$$

Example 2. If $f(z) = C_0(z-z_0)^{-1}$, then

$$\begin{cases} u = \frac{C_0(y_0-y)}{(x-x_0)^2+(y-y_0)^2}; \ v = \frac{C_0(x-x_0)}{(x-x_0)^2+(y-y_0)^2}, \\ P = [F - \frac{0{,}5C_0{}^2}{(x-x_0)^2+(y-y_0)^2} + C]\mu \end{cases} D = OXY\backslash\{M(x_0; y_0)\}. \tag{10}$$

In solutions (9) and (10) C and C_0 are arbitrary constants.

In [7] the solutions (10) with the constant $C_0 = 4(R_e)^{-1}$ are obtained by a more complicated method. They represent the flow velocities of a liquid and the pressure in the vicinity of the orifice located at the point $M(x_0; y_0)$.

3. Solutions, where the velocity components are solutions of Laplace's equation of a special form

Theorem. 2. If $w(x; y)$ is a harmonic function, i.e., $\Delta w = w_{xx} + w_{yy} = 0$, and has continuous partial derivatives up to the second order in the domain D, then system (1) admits the following solutions in this domain:

$$\begin{cases} u = w_y + C_1y + C_2x + C_4; v = -w_x + C_3x - C_2y + C_5, \\ P = [F - 0{,}5(u^2 + v^2) + (C_1 - C_3)[w + 0{,}5(C_1y^2 - C_3x^2) + C_2xy + C_4y - C_5x) + C]\mu \end{cases} \tag{11}$$

where C_k, $k = 1,...,5$ and C are arbitrary constants.

Demonstration T. 2. Suppose that $w(x; y)$ is a harmonic function and has inside the domain D continuous partial derivatives up to and including the second order. And let the functions u and v have the form in (11). Then $u_x + v_y = 0; u_y - v_x = C_1 - C_3$.
From here it follows that these functions verify equations (6) and (7), so they are solutions of system (1). It remains to find out the pressure. In this case we also have, that $\Delta u = 0, \ \Delta v = 0$.
From (5) we obtain:

$$\begin{cases} G_x = (w_x - C_3x + C_2y - C_5)(C_1 - C_3) \\ G_y = (w_y + C_1y + C_2x + C_4)(C_1 - C_3) \end{cases} \Rightarrow$$
$$G = (C_1 - C_3)[w + 0{,}5(C_1y^2 - C_3x^2) + C_2xy + C_4x - C_5y)] + C.$$

We substitute this result into (4) and obtain the pressure expression from (11). **T. 2.** is demonstrated.

Let's take an example of application of theorem 2.

Example 3. Let $w = ln(x^2 + y^2)$, $C_1 = 2$, $C_3 = 1$, $C_2 = C_4 = C_5 = 0$, then we obtain these solutions:

$$\begin{cases} u = \frac{2y}{x^2+y^2} + 2y; \;\; v = x - \frac{2x}{x^2+y^2}, \\ P = [F - \frac{4(2y^2-x^2+1)}{x^2+y^2} + ln(x^2+y^2) + y^2 - 0{,}5x^2]\mu + C \end{cases} \quad D = OXY \backslash \{O\}. \tag{12}$$

Note. If $C_1 \neq C_3$, then we obtain solutions of the form (11), which cannot be obtained from theorem 1. We mention that in the solutions obtained with the application of theorems 1 and 2 the viscosity parameter λ does not explicitly participate.

4. Solutions, in which the viscosity parameter participates explicitly. Method of separation of variables

Theorem 3. Let the functions $\varphi(x; y)$ and $z(x; y)$ admit in the domain D continuous partial derivatives up to the second order and let $T(\varphi)$ be a function doubly differentiable. Let these functions verify the following equations (13) and (14):

$$\left(\varphi_x^2 + \varphi_y^2\right) \cdot T'' + \Delta\varphi \cdot T' + \Delta z_0 = \varphi \tag{13}$$

$$\varphi_y \cdot z_x - \varphi_x \cdot z_y + \lambda\Delta\varphi = 0 \tag{14}$$

where z_0 is a particular solution of the equation (14).

The solutions for system (1) are determined in the following manner:

First on the determinates the functions u and v out from the following system:

$$\begin{cases} u = \varphi_y \cdot T' + z_{0_y} \\ v = -\varphi_x \cdot T' - z_{0_x} \end{cases} \tag{15}$$

then G from

$$\begin{cases} G_x = \lambda\varphi_y - v \cdot \varphi \\ G_y = -\lambda\varphi_x + u \cdot \varphi \end{cases} \tag{16}$$

and, finally, pressure P from the equality

$$P = \mu[G + F - 0{,}5(u^2 + v^2)]. \tag{17}$$

Demonstration T.3 We note $\quad u_y - v_x = \varphi(x; y). \quad$ (18)

Then from (7) and (18) we obtain that $\quad \Delta u = \varphi_y, \; \Delta v = -\varphi_x.$

Replacing in (6), we obtain for φ the following equation:

$$\lambda\Delta\varphi - u\varphi_x - v\varphi_y = 0 \tag{19}$$

In order to make sure that the condition (4) is fulfilled, we introduce the auxiliary function z, which has continuous partial derivatives of the second order in such a way that the equalities are true:

$$u = z_y; \; v = -z_x \tag{20}$$

We substitute expressions (20) in equation (19) and obtain equation (14).

Considering that φ is a given function, we can consider equation (14) in relation to z as a linear equation with partial derivatives of the first order [10]. It is easy to verify that the general solution of this equation has the form

$$z = T(\varphi) + z_0 \tag{21}$$

where $T(\varphi)$ is an arbitrary doubly differentiable function, and z_0 is a particular solution of equation (14). Substituting (21) into equalities (20) and then into (18), we obtain equation (13). Thus, finding the functions u, v and P is reduced to determining the functions T, φ and z from equations (13) and (14). **T. 3.** is demonstrated.

Note. For $\varphi \neq C$ – constant, theorem 3 generates a series of new solutions of the Navier-Stokes equations (1), which differ from those obtained in theorems 1 and 2; in this case at least one of the velocity components u or v will already not be a solution of Laplace's.

Relations (13) and (14) represent two equations with three unknown functions, which allow us to choose one of them in a convenient way for further study, and to determine the other two.

In the following we will study different particular cases, choosing in particular a certain form of the function φ.

Case 1. The function φ is a solution of Laplace's equation, that is $\Delta\varphi = 0$.

In this case, the general solution of equation (14) is $z = T(\varphi)$, $z_0 = 0$, and T is determined from the equation $(\varphi_x^2 + \varphi_y^2) \cdot T''(\varphi) = \varphi$

This equation can be solved when the expression $\varphi_x^2 + \varphi_y^2$ represents a function that depends on the variable φ or this expression is constant.

We note that in this case only the pressure P depends on the viscosity parameter λ, the functions u and v do not depend on the viscosity.

Example 4. May $\varphi = C\,ln(x^2 + y^2)$; $\varphi_x^2 + \varphi_y^2 = \frac{4C^2}{x^2+y^2} = \frac{4C^2}{e^{\varphi/C}}$; $T''(\varphi) = \frac{\varphi \cdot e^{\varphi/C}}{4C^2} \Rightarrow$

$$\Rightarrow T' = \frac{e^{\varphi/C}(\varphi - C)}{4C} + C_1 = \frac{(x^2 + y^2)[ln(x^2 + y^2) - 1]}{4} + C_1.$$

From (15) we obtain that

$$\begin{cases} u = \varphi_y \cdot T' = Cy \cdot [\frac{ln(x^2+y^2)-1}{2} + \frac{2C_1}{x^2+y^2}], \\ v = -\varphi_x \cdot T' = -Cx \cdot [\frac{ln(x^2+y^2)-1}{2} - \frac{2C_1}{x^2+y^2}]. \end{cases} \tag{22}$$

where $C \neq 0$ and C_1 are arbitrary constants. To determine the pressure according to the formula (17), we first find the function G from the system (16):

$$\begin{cases} G_x = \frac{2\lambda Cy}{x^2+y^2} + x \cdot Q; \;\; G_y = -\frac{2\lambda Cx}{x^2+y^2} + y \cdot Q; Q = C^2 \ln(x^2 + y^2)[\frac{ln(x^2+y^2)-1}{2} + \frac{2C_1}{x^2+y^2}]. \end{cases}$$

$$\Rightarrow$$

$$\begin{cases} G = -2\lambda C \cdot arctg(\frac{y}{x}) + \frac{C^2(x^2+y^2)}{4}[ln^2(x^2 + y^2) - 3\,ln(x^2 + y^2) + 3] + C_1\,ln(x^2 + y^2) + C_2 \\ P = \mu[G + F - 0{,}5(u^2 + v^2)] \end{cases}$$

Example 5. $\varphi_x^2 + \varphi_y^2 = C^2 -$ constant. We look for function φ in the following way: .

$\varphi = \alpha(x) + \beta(y) \Rightarrow \varphi_x^2 + \varphi_y^2 = (\alpha')^2 + (\beta')^2 = C^2 \Rightarrow (\alpha')^2 = C^2 - (\beta')^2 = C_1^2$. Then

$$\begin{cases} \alpha'(x) = \pm C_1 \\ \beta'(y) = \pm\sqrt{C^2 - C_1^2} \end{cases} \Rightarrow \begin{cases} \alpha = \pm C_1 x + c_1 \\ \beta = \pm\sqrt{C^2 - C_1^2}\,y + c_2 \end{cases} \Rightarrow \varphi = C_1 x + C_2 y + c, C_1^2 + C_2^2 = C^2.$$

Obviously, $\Delta\varphi = 0$. From $(\varphi_x^2 + \varphi_y^2) \cdot T'' = \varphi$ we obtain that $T'' = C^{-2}\varphi \Rightarrow T' = \frac{\varphi^2}{2C^2} + k$.

For the velocity components we get: $\begin{cases} u = \frac{C_2\varphi^2}{2C^2} + kC_2 \\ v = -\frac{C_1\varphi^2}{2C^2} - kC_1 \end{cases}$; $\varphi = C_1x + C_2y + c$. (23),

were $C^2 = C_1^2 + C_2^2$; c, C_1, C_2, k − arbitrary constants. From system (16) we find function *G*:

$$\begin{cases} G_x = \lambda C_2 + \dfrac{C_1\varphi^3}{2C^2} + kC_1\varphi \\ G_y = -\lambda C_1 + \dfrac{C_2\varphi^3}{2C^2} + kC_2\varphi \end{cases} \Rightarrow G = \lambda(C_2x - C_1y) + \frac{\varphi^4}{8C^2} + k\frac{\varphi^2}{2} + C_3.$$

The Pressure will be equal to $P = \mu[F + \lambda(C_2x - C_1y) + C_4]$, $C_4 = C_3 - \frac{(kC)^2}{2}$. (24)

Case 2. $\varphi = f(y) + g(x)$, where functions f and g are doubly differentiable. Then equation (14) has the following form:

$$f'(y)z_x - g'(x)z_y + \lambda[f''(y) + g''(x)] = 0 \Rightarrow \frac{z_x}{g'} - \frac{z_y}{f'} + \lambda\left(\frac{f''}{f'\cdot g'} + \frac{g''}{g'\cdot f'}\right) = 0. \quad (25)$$

We determine functions *f*(*y*) and *g*(*x*) in the following way: $\begin{cases} f''(y) = C_1f'(y) \\ g''(x) = C_2g'(x) \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} f'(y) = C_1f(y) + \alpha_1 \\ g'(x) = C_2g(x) + \alpha_2 \end{cases} \Rightarrow \begin{cases} f = k_1e^{C_1y} - \dfrac{\alpha_1}{C_1} \\ g = k_2e^{C_2x} - \dfrac{\alpha_2}{C_2} \end{cases} \Rightarrow \quad \varphi = k_1e^{C_1y} + k_2e^{C_2x} + \alpha \quad (26),$$

where k_1, k_2, α, C_1, C_2 are arbitrary constants.

A particular solution of equation (25) is $z_0 = \lambda(C_2y - C_1x)$.

Replacing function z_0 and φ from (26) in (13) we obtain the equation for determining function *T*:

$$(C_1^2k_1^2e^{2C_1y} + C_2^2k_2^2e^{2C_2x})T'' + (C_1^2k_1e^{C_1y} + C_2^2k_2e^{C_2x})T' = k_1e^{C_1y} + k_2e^{C_2x} + \alpha. \quad (27)$$

Equation (27) can take place when *T* is linear or one of the functions *f* or *g* is constant. May it be

$T = C\varphi \Rightarrow T' = C \Rightarrow T'' = 0$. Substituting in (27), we get first

$$(C_1^2k_1e^{C_1y} + C_2^2k_2e^{C_2x})C = k_1e^{C_1y} + k_2e^{C_2x} + \alpha \Rightarrow \alpha = 0, \; C_1^2 = C_2^2 = C^{-1}.$$

Replacing in (15) on obtain following solutions of system (1):

$$\begin{cases} u = k_1C_1^{-1}e^{C_1y} + \lambda C_2 \\ v = -k_2C_2^{-1}e^{C_2x} + \lambda C_1 \end{cases}; \; C_1^2 = C_2^2. \quad (28)$$

Then we determine the function G from the system (16):

$$\begin{cases} G_x = \lambda\varphi_y - v\cdot\varphi = \lambda k_1C_1e^{C_1y} + (k_2C_2^{-1}e^{C_2x} - \lambda C_1)(k_1e^{C_1y} + k_2e^{C_2x}) \\ G_y = -\lambda\varphi_x + u\cdot\varphi = -\lambda k_2C_2e^{C_2x} + (k_1C_1^{-1}e^{C_1y} + \lambda C_2)(k_1e^{C_1y} + k_2e^{C_2x}) \end{cases} \Rightarrow$$

$$\Rightarrow G = \frac{C(k_1e^{C_1y} + k_2e^{C_2x})^2}{2} + \lambda\left(\frac{k_1C_2e^{C_1y}}{C_1} - \frac{k_2C_1e^{C_2x}}{C_2}\right), C_2 = \pm C_1.$$

Then we find the pressure:

$$P = \mu[F + \lambda k_1 k_2 e^{C_1 y + C_2 x} + C_0]. \quad (29)$$

Let now that *f*(*y*) = *m* - constant. Then *φ* = *m* + *g*(*x*) and from (19) we obtain that:

$$-g'(x)z_y + \lambda g''(x) = 0 \quad \Rightarrow \quad u = z_y = \frac{\lambda g''(x)}{g'(x)};\ u_y = 0.$$

From (7) we have that

$$u_x + v_y = 0 \ \Rightarrow\ v = -u_x \cdot y + \beta(x).$$

From (18) it follows that $u_y - v_x = \varphi \ \Rightarrow\ v_x = -\varphi \ \Rightarrow\ -u_{xx} \cdot y + \beta'(x) = -m - g(x)$.

The last equality can only occur if $u_{xx} = 0,\ \beta'(x) = -m - g(x)$. From this results that

$$\left(\frac{g''(x)}{g'(x)}\right)'' = 0 \Rightarrow \frac{g''(x)}{g'(x)} = ax + b \Rightarrow g(x) = c\int e^{\left(\frac{ax^2}{2} + bx\right)}dx;\ \beta(x) = -mx - \int g(x)dx.$$

In this way we obtain the following solutions of the velocity components system (1):

$$u = \lambda(ax + b);\ v = -\lambda ay - mx - \int g(x)dx;\ g(x) = \int c\, e^{\left(\frac{ax^2}{2} + bx\right)}dx. \quad (30)$$

where *a*, *b*, *m*, *c* are arbitrary constants. The presure in this caze is:

$$P = \mu[F - 0{,}5a\lambda^2(x^2 + ay^2 + bx) - \lambda cy + C_0]. \quad (31)$$

If in (30) we take *a* = 0, then $\varphi = \frac{c}{b}e^{bx} + m$ we obtain the following solutions of system (1): $u = \lambda b;\ \ v = c_1 - mx - cb^{-2}e^{bx};\ \ P = \mu[F - \lambda(bm + c)y + C_0]$. (32)

If *a* = 0 and *b* = 0, then *g*(*x*) = c_1 + *cx* and we obtain the solutions:

$$u = 0,\ v = -\frac{c}{2\lambda}x^2 + c_1 x + c_2,\ P = \mu[F - \lambda cy + c_0]. \quad (33)$$

Le t *g*(*x*) = *m* - constant. Then *φ* = *m* + *f*(*y*) and we obtain:

$$v = \lambda(ay + b);\ u = \lambda ax + my + \int f(y)dy;\ f(y) = \int c\, e^{\left(\frac{ay^2}{2} + by\right)}dy. \quad (34)$$

Specifically, if in (33) we take *a* = 0, then $\varphi = \frac{c}{b}e^{by} + m.$ And we obtain solutions:

$$v = \lambda b;\ u = c_1 - my - \frac{c}{b^2}e^{by};\ \ P = [F - \lambda(bm + c)x + C_0]. \quad (35)$$

If *a* = 0, *b* = 0, then *f*(*y*) = *cy* + c_1 and we obtain the following solutions:

$$v = 0;\ u = -\frac{c}{2\lambda}y^2 + c_1 y + c_2;\ P = \mu(F - \lambda cx + c_0). \quad (36)$$

Case 3. *φ* = *f*(*y*)•*g*(*x*), where functions *f* and *g* are doubly differentiable.

In this case equation (14) is: $f'(y)g(x)z_x - g'(x)f(y)z_y + \lambda[g(x)f''(y) + f(y)g''(x)] = 0 \Rightarrow$

$$\frac{gz_x}{g'} - \frac{fz_y}{f'} + \lambda\left(\frac{gf''}{f' \cdot g'} + \frac{fg''}{g' \cdot f'}\right) = 0 \quad (37).$$

We determine f and g in the following way: $\begin{cases} f'(y) = C_1 f(y) \\ g'(x) = C_2 g(x) \end{cases} \Rightarrow \begin{cases} f'' = C_1 f' \\ g'' = C_2 g' \end{cases} \Rightarrow$

$$\begin{cases} f = \alpha_1 e^{C_1 y} \\ g = \alpha_2 e^{C_2 x} \end{cases} \Rightarrow \ \varphi = \alpha e^{C_1 y + C_2 x}, \alpha = \alpha_1 \alpha_2 \quad (38)$$

Then from (34) we obtain:

$$\frac{z_x}{C_2} - \frac{z_y}{C_1} + \lambda\left(\frac{C_1}{C_2} + \frac{C_2}{C_1}\right) = 0 \;\Rightarrow\; C_1 z_x - C_2 z_y = -\lambda(C_1^2 + C_2^2) \tag{39}$$

The general solution of equation (38) is $z = T(\varphi) + z_0$, where $z_0 = \lambda(C_2 y - C_1 x)$.

Substituting (38) in (13), we obtain the equation for $T(\varphi)$:

$$[C_1^2\alpha^2 e^{2(C_1 y + C_2 x)} + C_2^2\alpha^2 e^{2(C_2 x + C_1 y)}]T'' + [\alpha(C_1^2 + C_2^2)e^{C_2 x + C_1 y}]T' = \alpha e^{C_1 y + C_2 x}.$$

From here $$\alpha e^{(C_2 x + C_1 y)} \cdot T'' + T' = \frac{1}{(C_1^2 + C_2^2)} \tag{40}$$

Or $$\varphi \cdot w' + w = m;\; m = \frac{1}{(C_1^2 + C_2^2)},\; w(\varphi) = T'(\varphi). \tag{41}$$

Equation (40) is a linear ordinary differential equation of order 1 with the unknown function *w*. The general solution ([11], [12]) of the given equation is: $w = m + \frac{c}{\varphi} = F'$. In this case we obtain the following solutions of system (1):

$$\begin{cases} u = m_1 e^{C_1 y + C_2 x} + cC_1 + \lambda C_2 \\ v = -m_2 e^{C_1 y + C_2 x} - cC_2 + \lambda C_1 \end{cases};\; m_n = \frac{\alpha C_n}{C_1^2 + C_2^2},\; n = 1, 2.$$

$$G = 0{,}5\alpha^2 m e^{2(C_1 y + C_2 x)} + \alpha c e^{(C_1 y + C_2 x)} + C_0;\quad P = \mu[\,F + (\lambda^2 + c^2)(C_1^2 + C_2^2)]. \tag{42}$$

Where α, c, C_1, C_2 are arbitrary constants.

<u>Note.</u> In the case of the current given by (41), in the absence of external force, the pressure is constant. Solutions (41) and (23) represent those cases, when there is a linear dependence between the velocity components. Thus, in the case of these solutions we have correspondingly, that

$$v = -\frac{C_2}{C_1}u + \lambda C_1 + \frac{\lambda C_2^2}{C_1};\; v = -\frac{C_1}{C_2}u.$$

Case 4. $\varphi = f(t) = f(kx + by)$, where function $f(t)$ is doubly differentiable.

In this case equation (14) has the following form:

$$bz_x - kz_y + \lambda(b^2 + k^2)\frac{f''}{f'} = 0 \tag{43}$$

We will examine the case $\frac{f''}{f'} = \mathrm{C} - \text{constant},\; C \neq 0.$ Then

$$f'(t) = C_1 e^{Ct} \Rightarrow \varphi = f(t) = \left(\frac{C_1}{C}\right)e^{Ct} + C_2;\; t = kx + by. \tag{44}$$

The general solution for equation (42) will be $z = T(t) + z_0$, where $z_0 = \lambda C(ky - bx)$,

From (6) we obtain the equation for determining the function T:

$$f'T'' + CT' = \frac{f}{f'(k^2 + b^2)}, f' = C(f - C_2). \tag{45}$$

Or $$(f - C_2) \cdot w' + w = \frac{f}{m(f - C_2)};\; m = C^2(k^2 + b^2),\; w(f) = T'(f). \tag{46}$$

We obtained a linear ordinary differential equation of order 1 with the unknown function *w*.

The general solution for equation (46) is $w = \frac{1}{m(f - C_2)}[f + C_2 \ln(f - C_2) + C_0]$.

Hence, considering relations (44) and (45), we obtain the following equality:

$$T' = \frac{e^{-Ct}}{mC_1}[C_1 e^{Ct} + C_2 C^2 t + C_3], C_3 = C_0 + CC_2 ln\left(\frac{C_1}{C}\right). \tag{47}$$

In this case for the velocity components we have the following solutions:

$$\begin{cases} u = \varphi_y \cdot T' + z_{0_y} = \frac{b}{m}[C_1 e^{Ct} + C_2 C^2 t + C_3] + \lambda k C \\ v = -\varphi_x \cdot T' - z_{0_x} = -\frac{k}{m}[C_1 e^{Ct} + C_2 C^2 t + C_3] + \lambda b C \end{cases}; \ t = kx + by. \quad (48),$$

where $m = C^2(k^2 + b^2)$; k, b, C, C_1, C_2, C_3 are arbitrary constants.

Note. If in (48) we take $C_2 = 0$, then we will obtain solutions of the form (42).

Case 5. $\Delta\varphi = C_1\varphi_x + C_2\varphi_y$, with $C_1, C_2 -$ constants.

In this case equation (14) has the following form:

$$\frac{z_x}{\varphi_x} - \frac{z_y}{\varphi_y} + \lambda\left(\frac{C_1}{\varphi_y} + \frac{C_2}{\varphi_x}\right) = 0. \quad (49)$$

The general solution of this equation will be $z = T(f) + z_0$, where

$z_0 = \lambda(C_1 y - C_2 x)$.

We will determine the function φ, applying the method of separation of variables ([12], [13]).

If $\varphi = a(x) + b(y)$, then we will obtain solutions of the form (S_{33}) – (S_{35}) examinate of case 2.

Let $\varphi = a(x)b(y)$. Then from $\Delta\varphi = C_1\varphi_x + C_2\varphi_y \Rightarrow \frac{a''}{a} + \frac{b''}{b} = C_1\frac{a'}{a} + C_2\frac{b'}{b} \Rightarrow$ $\frac{a'' - C_1 a'}{a} = \frac{-b'' + C_2 b'}{b} = C$, where C is an arbitrary constant.

From here, to determine the functions $a(x)$ and $b(y)$ we obtain the following system of ordinary differential equations:

$$\begin{cases} a'' - C_1 a' - Ca = 0, \\ b'' - C_2 b' + Cb = 0. \end{cases} \quad (50)$$

The general solution of the first equation in (50), depending on the value of $k_1 = C_1^2 + 4C$, will be:

$$1)\ k_1 > 0;\ a(x) = a_1 e^{\alpha_1 x} + a_2 e^{\alpha_2 x}, \qquad \alpha_{1,2} = 0{,}5(C_1 \pm \sqrt{k_1});$$

$$2)\ k_1 < 0;\ a(x) = e^{0{,}5C_1 x}\left(a_1\cos\left(\sqrt{-k_1}x\right) + a_2\sin\left(\sqrt{-k_1}x\right)\right);$$

$$3)\ k_1 = 0;\ a(x) = e^{0{,}5C_1 x}(a_1 + x a_2).$$

For the second equation in (50) the form of the general solution will be the same, but depending on the value of $k_2 = C_2^2 - 4C$, replacing in 1), 2), 3) x with y, $\alpha_{1,2}$ with $\beta_{1,2}$ and the arbitrary constants a_1, a_2, with the corresponding arbitrary constants b_1, b_2 ([10], [11]).

From (13) we deduce the equation for determining function T:

$$[(a'b)^2 +\cdot (b'a)^2]F'' +\cdot [a''b + b''a]F' = ab. \quad (51)$$

We will first consider that $T' = l$ - constant. Then $T'' = 0$ and from (51) we obtain that

$$[\frac{a''}{a} + \frac{b''}{b}]l = 1. \quad (52)$$

Equality (52) can only occur if both fractions in the square bracket have constant values. We will examine these possibilities.

a) $-C_1^2 < 4C < C_2^2$. Then $k_1 > 0$ și $k_2 > 0$ and we take $a(x) = a_n e^{\alpha_n x}, b(y) = b_m e^{\beta_m y}$; $n, m = 1, 2,$ where $\alpha_{1,2} = 0{,}5\left(C_1 \pm \sqrt{k_1}\right)$, $\beta_{1,2} = 0{,}5\left(C_2 \pm \sqrt{k_2}\right)$, Then $a'' = \alpha_n^2 a,\ b'' = \beta_m^2 b$ și $l = 1/(\alpha_n^2 + \beta_m^2)$. In this case we obtain the following solutions of system (1):

$$\begin{cases} u = a_n b_m \beta_m e^{\alpha_n x} e^{\beta_m y} \cdot l + \lambda C_1,\ l = 1/(\alpha_n^2 + \beta_m^2) \\ v = -a_n b_m \alpha_n e^{\alpha_n x} e^{\beta_m y} \cdot l + \lambda C_2 \qquad n, m = 1, 2 \end{cases}. \tag{53}$$

In this case the pressure will be $P = \mu[\,F + \lambda^2(C_1^2 + C_2^2) + C_0]$.

b) $C_2 = 0, C_2^2 < 4C$. Then $k_1 > 0, k_2 < 0$ and $a(x) = a_n e^{\alpha_n x}, \alpha_{1,2} = 0{,}5\big(C_1 \pm \sqrt{k_1}\,\big)$,
$b(y) = b_1 cos\sqrt{C}y + b_2 sin\sqrt{C}y\,;$ $a_n\,, b_m$ are arbitrary constants; $n, m = 1, 2$.

Then $a'' = \alpha_n^2 a,\ b'' = -Cb$ și $l = 1/(\alpha_n^2 - C)$. It is easy to verify that if the constants that if the constants C and C_1 are different from zero, then also $\alpha_n^2 - C \neq 0$. In this case we obtain the following solutions:

$$\begin{cases} u = a_n \sqrt{C} e^{\alpha_n x}(b_2 cos\sqrt{C}y - b_1 sin\sqrt{C}y) l + \lambda C_1,\ l = 1/(\alpha_n^2 - C) \\ \quad v = -a_n \alpha_n e^{\alpha_n x}(b_1 cos\sqrt{C}y + b_2 sin\sqrt{C}y) l \qquad n = 1, 2 \end{cases}. \tag{54}$$

c) $C_1 = 0$ și $4C < -C_1^2$. Then $k_1 < 0, \quad k_2 > 0$ and $a(x) = a_1 cos\sqrt{-C}x + a_2 sin\sqrt{-C}x$,
$b(y) = b_m e^{\beta_m y};\ \beta_{1,2} = 0{,}5\big(C_2 \pm \sqrt{k_2}\,\big); a_n, b_m -$ arbitrary const.$; n, m = 1, 2$

Then $a'' = Ca,\ b'' = \beta_m^2 b$ and $l = 1/(C + \beta_m^2)$. It is easy to verify that if C and C_2 are different from zero then also $C + \beta_m^2 \neq 0$. In this case we obtain the following solutions:

$$\begin{cases} \quad u = b_m \beta_m e^{\beta_m y}(a_1 cos\sqrt{-C}x + a_2 sin\sqrt{-C}x) l\ ,\ l = 1/(C + \beta_m^2) \\ v = -b_m \sqrt{-C} e^{\beta_m y}\big(a_2 cos\sqrt{-C}y - a_1 sin\sqrt{C}y\big) l + \lambda C_2; \qquad n = 1, 2 \end{cases}. \tag{55}$$

In case the solutions (54) and (55) on determine the function *G* with the (16) and the pressure *P* with the formula (17).

Note. The case $k_1 < 0$ and $k_2 < 0$ is impossible because it only takes place if $C_2^2 < 4C < -C_1^2$.
In the case $k_1 = 0$ and $k_2 = 0$ we obtain solutions (53) with $a_n = 0{,}5C_1,\ b_m = 0{,}5C_2$.
If $\alpha_2^2 = \alpha_1^2$ and $\beta_2^2 = \beta_1^2$ then $C_1 = C_2 = 0$. But in this case we have $a'' = Ca, b'' = -Cb$ and the expression from the square brackets of the equation (52) is equal to zero, that is (52) becomes a false identity.
We now return to equation (51) in the case where *T'* is not constant. In this case we proceed in the same way as in case a) above and take $a(x) = a_n e^{\alpha_n x}, b(y) = b_m e^{\beta_m y}$. We have the following equation for the determination of the function *T*:

$$a_n b_m e^{\alpha_n x + \beta_m y} \cdot T'' + T' = \frac{1}{(\alpha_n^2 + \beta_m^2)}.$$

This equation has the form of equation (40) studied previously. Solving it in the same way as in the case of equation (40) we will obtain solutions of the form (42), replacing C_1 with α_n, C_2 with β_m and $\alpha = a_n b_m$.

5. Results and Discussion

The present paper contains three proven theorems, as well as various examples and particular examined cases. Applying Theorem 1, we can find various solutions, where the velocity components represent the imaginary and real parts of a differentiable function of a complex variable. Theorem 2 allows us to determine solutions, where the velocity components are expressed by the partial derivatives of the solutions of Laplace's equation of a special form. It is to be mentioned that these theorems give us solutions that do not depend on the viscosity parameter *λ*.

In the present work, an original method elaborated by the author is presented, with the help can be determined of which solutions of the system (1), in which the viscosity parameter participates explicitly. This method is presented in theorem 3 for obtaining a series of solutions of the Navier-Stokes equations in which the viscosity coefficient λ participates explicitly. The application of this method to obtain different solutions of system (1) is presented in cases 1-5 following the proof of theorem 3.

In the current article a lot of solutions of the Navier-Stokes equations are determined by the system (1), both of general form - solutions (2), (11), as well as the exact solutions - (9), (10), (12), (22), (23), (24), (28), (29), (30)-(36), (42), (48), (52)-(55). Starting with the solution (28) onwards, in all the obtained solutions, their dependence on the viscosity parameter λ is explicitly indicated. We would also like to mention that the main results presented in this paper were reported and discussed in the following international conferences: the symposium UTM 2020, conference MITRE 2021 and CAIM 2022. The obtained results were presented in the following theses of these conferences: [14], [15] and [16].

6. Conclusions

In this paper a lot of solutions of the Navier-Stokes equations are determined both of general form as well as the exact solutions. Different constants participate in the expressions of the obtained solutions, the values of which can be determined based on the initial conditions and the boundary conditions of the examined physical problems.

For example, the solutions (33) or (36) are solutions of the plane flow problem type Poiseuille or Couette ([17]). Thus, for the plane flow of Poiseuille type, we have that for the channel section of diameter *2h*, a component of the velocity is equal to zero, for example v = 0, and the boundary conditions are of the form $u(h) = u(-h) = 0$. From here we find that $c_1 = 0$, $c_2 = -(h^2c)/2\lambda$, and $c > 0$, since the pressure inside the channel decreases. For the Couette type flow we also have $v = 0$ and the following boundary conditions: $u(h) = u_0$, $u(-h) = 0$. Then $c_1 = u_0/2h$,
$c_2 = (h^2c)/2\lambda + u_0/2$ ([17]).

In the research carried out here, the rotor (vortex) of the examined flow is of special importance, namely the expression $\varphi = u_y - v_x$. Thus, if we can experimentally or theoretically determine the rotor of the flow (is it null, or constant, or has a particular shape), then we can apply the results of theorems 1, 2, or 3, demonstrated above, to determine the components of the flow velocity and the pressure created.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cernica, I. *Fluid mechanics*. Matrixrom, București, 2011, 425 p. [in Romanian].
2. Mazo, A.; Potasev, K. *Hydrodynamics*. CGU, 2013, 128 p. [in Russian].
2. Loitseanschii, L. *Mechanics of fluids and gases*. Nauka, Moscow, 2003, 840 p. [in Russian].
4. Wang, C. Exact solution of the unsteady Navier-Stokes equations. *Appl. Mech. Rev.* 1989, 42, pp. 269-282.
5. Wang, C. Exact solution of the steady-state Navier-Stokes equations. *Annu. Rev. Fluid Mech. Rev.* 1991, 23, pp. 159-177.
6. Polyanin, A.; Zaitsev, V. Exact solutions of the Navier-Stokes equations with generalized separation of variables. *Dokladi RAN* 2001, 380, pp. 491-496. [in Russian]
7. Koptev, A. First integral and ways of further integration of the Navier-Stokes equations. *Izvestia RGU Hertsen, N.* 2012, 147, p. 7-17. [in Russian].

8. Polyanin, A.; Zaitsev, V. *Handbook of nonlinear partial differential equations*. Boca Raton, CRC Press, 2004, 840 p.
9. Enescu, I.; Sava, V. *Special mathematics*. Institutul Politehnic, Iasi, 1981, 391 p. [in Romanian].
10. Stepanov, V. *Differential equations course*. Ed. Lumina, Chisinau, 1970, 488 p. [in Romanian].
11. Kamke, E. *Handbook of Ordinary Differential Equations*. Nauka, Moscow, 1976, 576 p. [in Russian].
12. Aniculăesei, Gh. *Differential and partial differential equations*. Universității AIC, Iași, 2003, 375 p. [in Romanian].
13. Perjan, A. *Partial differential equations*. USM, Chisinau, 2015, 225 p. [in Romanian].
14. Baltag, Iu. On some determination solutions of the stationary bidimensional Navier-Stokes equations. In *International Conference SIPAMI 2020*, Chisinau, 27 – 28 november 2020, p. 46.
15. Baltag, Iu. Determination on some solutions of the stationary 2D Navier-Stokes equations. In *International Conference MITRE 2021*, Chisinau, 01 – 03 Jule 2021, p. 14.
16. Baltag, Iu. Determination of some solutions of the stationary Navier-Stokes equations. In *International Conference CAIM 2021*, Romania, 17 – 18 september 2021, p. 12.
17. Vedeneev, V. *Mathematical theory of stability of plane-parallel fluid flows*. ID Intelect, Moscow, 2016, 152 p. [in Russian].

Citation: Baltag, Iu. Determination of some solutions of the stationary 2D Navier-Stokes equations. *Journal of Engineering Science* 2022, 29(4), pp. 38-50. https://doi.org/10.52326/jes.utm.2022.29(4).03.

Publisher's Note: JES stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright:© 2022 by the authors. Submitted for possible open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

Submission of manuscripts: jes@meridian.utm.md