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SOLVING THE DAYS-OFF SCHEDULING PROBLEM USING QUADRATIC PROGRAMMING WITH CIRCULANT MATRIX

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Abstract. The purpose of this paper is the approach of a mathematical model dedicated to planning the consecutive days off of a company's employees. Companies must find a flexible work schedule between employees, always considering the satisfaction of work tasks as well as guaranteeing consecutive days off. The analysis is based on solving a quadratic programming problem with binary variables. The proposed method uses the properties of the circulant symmetric matrix in the researched model, which allows the transformation of the considered problems into an equivalent separable non-convex optimization problem. A practical continuous convex relaxation approach is proposed. DC Algorithm is used to solve relaxed problems. A solved numerical example is presented.

Keywords: *scheduling problem, binary quadratic programming, circulant matrix, separable optimization, convex relaxation.*

Rezumat. Scopul acestei lucrări este abordarea unui model matematic dedicat planificării zilelor libere consecutive ale angajaților unei companii. Companiile trebuie să găsească un program de lucru flexibil între angajați, având întotdeauna în vedere satisfacerea sarcinilor de lucru cât și garantarea zilelor libere consecutive. Tratarea se bazează pe rezolvarea unei probleme de programare pătratică cu variabile binare. Metoda propusă utilizează proprietățile matricei simetrice circulante din modelul cercetat, care permite transformarea problemei considerate într-o problemă echivalentă de optimizare neconvexă separabilă. Este propusă o abordare practică de relaxare continuă convexă. Pentru rezolvarea problemei relaxate se utilizează Algoritmul DC. Se prezintă un exemplu numeric rezolvat.

Cuvinte-cheie: *problemă de planificare, programare pătratică binară, matrice circulantă, optimizare separabilă, relaxare convexă.*

1. Introduction

In both production organizations and service companies (IT companies, airlines, security services, fire stations, restaurants, hospitals, etc.) a specific concern is the planning of days off and working days for a week (or multiples thereof). Work schedules should require employees to be present at work on different days to maintain the admissible quality of services. At the same time, employees must have the desired number of consecutive days off during the week. The adaptability for days-off planning in companies is a serious problem, especially in recent years, during the Covid-19 pandemic. Companies need to find a flexible work schedule between employees, always considering the satisfaction of work tasks as well as the guarantee of consecutive days off. A comprehensive study of the literature on this subject can be found in [1, 2].

In this paper, the mathematical model proposed in [3] is developed, in which the problem of free days planning is reformulated as a binary problem of non-convex quadratic programming. In the case of convex, effective (polynomial) solving algorithms have been proposed [4]. But if the quadratic programming problem is nonconvex or includes integer variables, the problem is NP-hard [5, 6]. No known algorithm can solve such problems efficiently.

The paper is organized as follows: Sections 3 and 4 describe the issue of day planning and formulated as a binary problem of quadratic programming. Section 5 investigates the properties of the symmetric matrix that form the purpose matrix. In Section 5 the quadratic programming problem turns into an equivalent separable programming problem.

2. Problem definition

We will consider that the working week is 5 days, i.e., the company or institution has a compressed and flexible work schedule. The days are marked to be 1, 2, 3, 4, 5. The planning of days off and working days per week or the multi-week work cycle assumes that the following assumptions and constraints are met [7, 3]:

1. The total number of employees m and the number of workers n_k required during the day are known: $k \in \{1, 2, 3, 4, 5\}$.
2. Each employee $i \in \{1, 2, \dots, m\}$ has a fixed number d_i of days off per week.
3. Each employee has at least 2 consecutive days off per week.
4. Some specific tasks need to be assigned to employees who have the skills to perform them.

3. The 0-1 Quadratic Programming Model [3]

For each worker, we enter the binary variable x_{ik} so that:

$$x_{ik} = \begin{cases} 1, \text{if day } k \in \{1,2,3,4,5\} \text{ is an off day for worker } i \in \{1,2,\dots,m\} \\ \qquad\qquad 0, \text{ otherwise} \end{cases}$$

We have the matrix (Eq. (1)):

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & x_{m3} & x_{m4} & x_{m5} \end{pmatrix} \in \mathbb{R}^{m\times 5} \tag{1}$$

We convert this matrix into a column vector consisting of its lines (Eq. (2)):

$$x = (x_{11} \quad x_{12} \quad \dots \quad x_{15} \quad x_{21} \quad x_{22} \quad \dots \quad x_{25} \quad x_{m1} \quad x_{m2} \quad \dots \quad x_{m5})^T. \tag{2}$$

For every day $k \in \{1,2,3,4,5\}$, the number of workers taking this day off is $m - n_k$. This assumption can be written as:

$$\begin{cases} x_{11} + x_{21} + \cdots + x_{m1} = m - n_1 \\ x_{12} + x_{22} + \cdots + x_{m2} = m - n_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{15} + x_{25} + \cdots + x_{m5} = m - n_5 \end{cases}.$$

Or in the matrix form

$$Ax=b$$

where $b = (m - n_1 \quad m - n_2 \quad m - n_3 \quad m - n_4 \quad m - n_5)^T \in \mathbb{R}^5$ and

$$A = (I \quad I \quad \dots \quad I) \in \mathbb{R}^{5\times(5m)}.$$

Here *I* is the 5×5 identity matrix.

The assumption that each worker (employee) $i \in \{1,2,\dots,m\}$ has d_i days off per week can be written as follows:

$$\begin{cases} x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = d_1 \\ x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = d_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{m1} + x_{m2} + x_{m3} + x_{m4} + x_{m5} = d_m \end{cases}$$

or *Ex=d* , where

$$d = (d_1 \quad d_2 \quad \dots \quad d_m)^T \in \mathbb{R}^m,$$
$$E = (E_1 \quad E_2 \quad \dots \quad E_m) \in \mathbb{R}^{m\times(5m)}.$$

The matrices $E_i \in \mathbb{R}^{m\times 5}$ are in the form

$$E_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow the\ row\ i \in \{1,2,\dots,m\}.$$

The assumption that some specific tasks must be performed by a worker with the appropriate skill can be written as follows:

$$c_{ik} = \begin{cases} 1, \text{if the presence of the worker } i \in \{1,2,\dots,m\} \text{ is needed for the day } k \in \{1,2,3,4,5\} \\ 0, \quad \text{otherwise} \end{cases}$$

Then

$$Cx=0,$$

where $0 = (0 \quad 0 \quad \dots \quad 0)^T \in \mathbb{R}^m$ – is the zero-column vector and $C = \{0,1\}^{m\times(5m)}$ is the matrix with the c_{ik} elements defined above.

We need a function that allows us to maximize the number of consecutive days off per week [3]:

$$f(x) = \sum_{i=1}^{m} \left(\sum_{k=1}^{4} x_{ik}x_{i,k+1} + x_{i5}x_{i1}\right).$$

The objective function *f(x)* can be written as follows:

$$f(x) = x^T Q x.$$

Here the symmetric matrix Q is a block diagonal matrix:

$$Q = \begin{pmatrix} Q_0 & O & \dots & O \\ O & Q_0 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & Q_0 \end{pmatrix} \in \mathbb{R}^{(5m)\times(5m)}.$$

Where zero matrix $O \in \mathbb{R}^{5\times 5}$ is a matrix in which all of the entries are 0, and

$$Q_0 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \in \mathbb{R}^{5\times 5}$$

Thus, maximizing the number of consecutive days off is to solve the following problem of quadratic programming with binary variables:

$$\left.\begin{array}{c} f(x) = x^T Q x \to max \\ \text{subject to} \\ Ax = b, \\ Ex = d, \\ Cx = 0, \\ x \in \{0,1\}^n \end{array}\right\} \qquad \text{(QP)}$$

When solving the problem (QP), we encounter great difficulties because the function is non-convex and the variables are integers (Boolean). The problem (QP) in general case is NP-hard [5, 8] and is very difficult to solve. The interest in such issues is the subject of a rich literature (see synthesis papers [9]) and has given rise to the development of numerous methods:

- Lagrangian relaxation to semi-defined relaxation [10, 11].
- Conical relaxation [12, 13].
- Continuous reformulation [14 - 16].
- Linear reformulation (reformulation-linearization technique RLT) [17].
- Heuristic and genetic methods [18, 19].

This paper proposes a method for solving the problem (QP), using the separable reformulation. This method is based on diagonalizing the Q matrix.

4. Q_0 matrix eigenvalues and eigenvectors

The matrix Q_0 is a symmetric circulant matrix [20]. It is immediately verified that the vector $x_p^{(1)} = (1 \quad 1 \quad 1 \quad 1 \quad 1)^T$ is an eigenvector. Indeed

$$Q_0 x_p^{(1)} = \left(\frac{1}{2} + \frac{1}{2}\right) x_p^{(1)} = x_p^{(1)}.$$

Thus $\lambda_1 = 1$ is an eigenvalue of the matrix Q_0. The other eigenvectors can be easily found using the primitive root of unity: $\omega_5 = exp(2\pi i/5)$, $i = \sqrt{-1}$.

In ω_5 terms, the the eigenvalues of the matrix Q_0 are [20]:

$$\lambda_k = \frac{1}{2}\omega_5^{k-1} - \frac{1}{2}\omega_5^{4(k-1)},\ k = 1,2,3,4,5.$$

So, we have (Eq. 3):

$$\left.\begin{aligned} \lambda_1 = 1\,,\ \lambda_2 = \frac{1}{4}\sqrt{5} - \frac{1}{4}\,,\ \lambda_3 = -\frac{1}{4}\sqrt{5} - \frac{1}{4} \\ \lambda_4 = -\frac{1}{4}\sqrt{5} - \frac{1}{4} = \lambda_3,\ \ \lambda_5 = \frac{1}{4}\sqrt{5} - \frac{1}{4} = \lambda_2 \end{aligned}\right\}. \tag{3}$$

The eigenvectors are determined using the same primitive root of the ω_5 unit:

$$v_p^{(k)} = \begin{pmatrix}\omega_5^{0\times(k-1)} & \omega_5^{1\times(k-1)} & \omega_5^{2\times(k-1)} & \omega_5^{3\times(k-1)} & \omega_5^{4\times(k-1)}\end{pmatrix}^T,\ k = 1,2,3,4,5.$$

We note (Eq. 4):

$$\left.\begin{aligned} \mu_2 = \frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}}, \\ \mu_3 = \frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \end{aligned}\right\}. \tag{4}$$

Performing the respective calculations and taking into account (Eq. 3 and Eq. 4), we obtain the eigenvectors of the matrix Q_0:

$$v_p^{(1)} = (1 \quad 1 \quad 1 \quad 1 \quad 1)^T = (\lambda_1 \quad \lambda_1 \quad \lambda_1 \quad \lambda_1 \quad \lambda_1)^T,$$

$$v_p^{(2)} = \begin{pmatrix} 1 \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 + i\mu_2 \\ \lambda_3 + i\mu_3 \\ \lambda_3 - i\mu_3 \\ \lambda_2 + i\mu_2 \end{pmatrix},$$

$$v_p^{(3)} = \begin{pmatrix} 1 \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_3 + i\mu_3 \\ \lambda_2 - i\mu_2 \\ \lambda_2 + i\mu_2 \\ \lambda_3 - i\mu_3 \end{pmatrix},$$

$$v_p^{(4)} = \begin{pmatrix} 1 \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_3 - i\mu_3 \\ \lambda_2 + i\mu_2 \\ \lambda_2 - i\mu_2 \\ \lambda_3 + i\mu_3 \end{pmatrix},$$

$$v_p^{(5)} = \begin{pmatrix} 1 \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} - i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \\ -\frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}} \\ \frac{1}{4}\sqrt{5} - \frac{1}{4} + i\frac{1}{4}\sqrt{2}\sqrt{5+\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 - i\mu_2 \\ \lambda_3 - i\mu_3 \\ \lambda_3 + i\mu_3 \\ \lambda_2 + i\mu_2 \end{pmatrix}.$$

Since Q_0 is a symmetric matrix with real elements and real eigenvalues, then we can always choose the corresponding eigenvectors with real inputs. Indeed, if

$$v = a + ib,\ ab \neq 0,\ i = \sqrt{-1},$$

is an eigenvector of the matrix *M* with eigenvalue λ. Then

$$M(a+ib) = \lambda(a+ib).$$

From here it follows that

$$Ma = \lambda a, Mb = \lambda b,$$
$$M(a+b) = \lambda(a+b),$$

i.e., the vectors *a, b* and $a+b$ are also eigenvectors associated with eigenvalue λ. Therefore, the real parts, the imaginary parts and their sum will in turn be their eigenvectors. Thus, we have the eigenvectors of the matrix Q_0 with the real elements:

$$p_k = Re\left(v_p^{(k)}\right) + Im(v_p^{(k)}),\ k = 1,2,3,4,5.$$

To establish other properties of the matrix Q_0 we consider the matrix whose columns are the eigenvectors p_k:

$$F_0 = \frac{1}{\sqrt{5}}(p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5) = \begin{pmatrix} \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 \\ \lambda_1 & \lambda_2+\mu_2 & \lambda_3+\mu_3 & \lambda_3-\mu_3 & \lambda_2-\mu_2 \\ \lambda_1 & \lambda_3+\mu_3 & \lambda_2-\mu_2 & \lambda_2+\mu_2 & \lambda_3-\mu_3 \\ \lambda_1 & \lambda_3-\mu_3 & \lambda_2+\mu_2 & \lambda_2-\mu_2 & \lambda_3+\mu_3 \\ \lambda_1 & \lambda_2-\mu_2 & \lambda_3-\mu_3 & \lambda_3+\mu_3 & \lambda_2+\mu_2 \end{pmatrix}.$$

The F_0 matrix is the Discrete Fourier Transform (DFT) matrix, with its orthogonal columns (orthonormal).

We will note the following properties of the F_0 matrix, properties that are verified by direct calculation:

1. $F_0 = F_0^T$
2. $F_0^2 = I$
3. $F_0^{-1} = F_0$
4. $\det(F_0) = 1$
5. $cond(F_0) = 1$

Thus, F_0 is a well-conditioned special orthogonal symmetric matrix. This is important from a numerical calculation point: small disturbances in the input data will not produce large variations in the calculations [21]. Using the orthogonal matrix F_0, the diagonalization of the circulating matrix Q_0 can be achieved (Eq. 5) [21]:

$$F_0 Q_0 F_0 = \Lambda_0 \tag{5}$$

where

$$\Lambda_0 = Diag(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5) = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix},$$

namely (Eq. (6)):

$$Q_0 = F_0 \Lambda_0 F_0 \tag{6}$$

5. A separable reformulation of the quadratic program (QP)

Consider now the direct sum *m* of the matrix Q_0: $Q_0 \oplus Q_0 \oplus \cdots \oplus Q_0 \oplus Q_0$, which as a result gives us the matrix Q. Matrix Q is a block circulant matrix. The eigenvalues and eigenvectors of the matrix Q are the sum of the sets of 5 eigenvectors $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and respectively the 5 eigenvectors p_1, p_2, p_3, p_4, p_5.

We note the diagonal block matrices:

$$F = F_0 \oplus F_0 \oplus \cdots \oplus F_0 \oplus F_0 = \begin{pmatrix} F_0 & O & \cdots & O & O \\ O & F_0 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & F_0 & O \\ O & O & \cdots & O & F_0 \end{pmatrix},$$

$$\Lambda = \Lambda_0 \oplus \Lambda_0 \oplus \cdots \oplus \Lambda_0 \oplus \Lambda_0 = \begin{pmatrix} \Lambda_0 & O & \cdots & O & O \\ O & \Lambda_0 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & \Lambda_0 & O \\ O & O & \cdots & O & \Lambda_0 \end{pmatrix}.$$

Then the matrix Q is diagonalized: $Q = F\Lambda F$.

Let be the column vector $\overline{x_i} = (x_{i1} \quad x_{i2} \quad x_{i3} \quad x_{i4} \quad x_{i5})^T$, *i*=1,2, ..., *n*.

Then the vector *x* defined in (Eq. 2) can be written:

$$x = (\bar{x}_1^T \quad \bar{x}_2^T \quad \cdots \quad \bar{x}_m^T)^T \in \mathbb{R}^{5m}.$$

With these notations we rewrite the objective function as follows:

$$f(x) = x^T Q x = \begin{pmatrix} \bar{x}_1^T & \bar{x}_2^T & \cdots & \bar{x}_{m-1}^T & \bar{x}_m^T \end{pmatrix} \begin{pmatrix} Q_0 & O & \cdots & O & O \\ O & Q_0 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & Q_0 & O \\ O & O & \cdots & O & Q_0 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{m-1} \\ \bar{x}_m \end{pmatrix} =$$

$$= \bar{x}_1^T Q_0 \bar{x}_1 + \bar{x}_2^T Q_0 \bar{x}_2 + \cdots + \bar{x}_{m-1}^T Q_0 \bar{x}_{m-1} + \bar{x}_m^T Q_0 \bar{x}_m,$$

or, taking into account (Eq. 6):

$$f(x) = (F_0 \bar{x}_1)^T \Lambda_0 F_0 \bar{x}_1 + (F_0 \bar{x}_2)^T \Lambda_0 F_0 \bar{x}_2 + \cdots + (F_0 \bar{x}_m)^T \Lambda_0 F_0 \bar{x}_m.$$

We determine the column vector

$$\bar{y}_i = F_0 \bar{x}_i = (y_{i1} \quad y_{i2} \quad y_{i3} \quad y_{i4} \quad y_{i5})^T,\ i=1, 2, 3, \dots, m,$$

with the elements

$$y_{ij} = \frac{\sqrt{5}}{5} p_j^T \bar{x}_i,\ i = 1,2,\dots,m;\ j = 1,2,3,4,5,$$

where $\frac{\sqrt{5}}{5} p_j$ is the column j of the matrix F_0.

We note

$$y = (\bar{y}_1^T \quad \bar{y}_2^T \quad \dots \quad \bar{y}_m^T)^T \in \mathbb{R}^{5m}.$$

We have (Eq. 7)

$$y = Fx = \begin{pmatrix} F_0 \bar{x}_1 \\ F_0 \bar{x}_2 \\ \vdots \\ F_0 \bar{x}_m \end{pmatrix}. \tag{7}$$

Then in the variables y_{ij} the purpose function becomes a separable function:

$$g(y) = \bar{y}_1^T \Lambda_0 \bar{y}_1 + \bar{y}_2^T \Lambda_0 \bar{y}_2 + \cdots + \bar{y}_m^T \Lambda_0 \bar{y}_m =$$

$$= \lambda_1 \sum_{i=1}^m y_{i1}^2 + \lambda_2 \sum_{i=1}^m y_{i2}^2 + \lambda_3 \sum_{i=1}^m y_{i3}^2 + \lambda_4 \sum_{i=1}^m y_{i4}^2 + \lambda_5 \sum_{i=1}^m y_{i5}^2 = \sum_{k=1}^5 \sum_{i=1}^m \lambda_k y_{ik}^2.$$

As $F^{-1} = F$, $F_0^{-1} = F_0$, from (Eq. 7), we have

$$x = Fy,\ \bar{x}_i = F_0 \bar{y}_i,\ i = 1,2,\dots,m,$$

$$x_{ik} = \frac{\sqrt{5}}{5} p_k^T \bar{y}_i,\ i = 1,2,\dots,m;\ k = 1,2,3,4,5.$$

Then the constraints

$$x_{1k} + x_{2k} + \dots + x_{mk} = m - n_k,\ k = 1,2,3,4,5,$$

$$x_{i1} + x_{i2} + x_{i3} + x_{i4} + x_{i5} = d_i,\ i = 1,2,3,\dots,m,$$

become (Eq. 8) and (Eq. 9):

$$p_k^T (\bar{y}_1 + \bar{y}_2 + \cdots + \bar{y}_m) = \sqrt{5}(m - n_k), \qquad k = 1,2,3,4,5, \tag{8}$$

$$(p_1 + p + p_3 + p_4 + p_5)^T \bar{y}_i = \sqrt{5} d_i,\ i = 1,2,\dots,m. \tag{9}$$

Considering

$$(p_1 + p_2 + p_3 + p_4 + p_5)^T = (5 \quad 0 \quad 0 \quad 0 \quad 0)^T,$$

from (Eq. 9) we obtain (Eq. 10):

$$y_{i1} = \frac{\sqrt{5}}{5} d_i, i = 1,2,\dots,m. \tag{10}$$

So

$$\bar{y}_i = (\tfrac{\sqrt{5}}{5} d_i \quad y_{i2} \quad y_{i3} \quad y_{i4} \quad y_{i5})^T, \; i = 1,2,\dots,m.$$

Thus, the problem (QP) was transformed into the following

$$g(y) = \sum_{k=1}^{5} \sum_{i=1}^{m} \lambda_k y_{ik}^2 \longrightarrow max \tag{11a}$$

subject to

$$p_k^T \left(\sum_{i=1}^{m} \bar{y}_i \right) = \sqrt{5}(m - n_i), \qquad k = 1,2,3,4,5, \tag{11b}$$

$$\left(\sum_{k=1}^{5} p_k^T \right) \bar{y}_i = \sqrt{5} d_i, i = 1,2,\dots,m, \tag{11c}$$

$$CFy = 0, \tag{11d}$$

$$p_k^T \bar{y}_i \in \{0, \sqrt{5}\} \; k = 1,2,3,4,5; \; i = 1,2,\dots,m. \tag{11e}$$

Problem (Eq. 11) is a non-convex separable quadratic programming problem, except for constraints (Eq. 11e). The object function (Eq. 11*a*) is a non-convex quadratic function in which $\lambda_1 > 0, \; \lambda_2 > 0, \; \lambda_3 = \lambda_4 < 0, \; \lambda_5 = \lambda_2 > 0$ This function can be rewritten as the difference between two convex functions:

$$g(y) = \varphi_1(y) - \varphi_2(y),$$

where:

$$\varphi_1(y) = \sum_{i=1}^{m} \lambda_1 y_{i1}^2 + \sum_{i=1}^{m} \lambda_2 y_{i2}^2 + \sum_{i=1}^{m} \lambda_5 y_{i5}^2,$$

$$\varphi_2(y) = \sum_{i=1}^{m} (-\lambda_3 y_{i3}^2) + \sum_{i=1}^{m} (-\lambda_4 y_{i4}^2).$$

Nowadays there is an original theory for problems with functions represented as the difference between two convex functions, problems called DC programming (DC-Difference of Convex Functions). For such problems, effective solving algorithms have been developed, called DCA (DC Algorithm) [9, 18]. Constraints (Eq. 11*b*), (Eq. 11*c*), and (Eq. 11*d*) are linear with the decision variables y_{ik}. The last constraint (Eq. 11*e*) forces the variables $\frac{\sqrt{5}}{5} p_k^T \bar{y}_i$ to take the value 0 or 1 (zero or one). This makes it very difficult to solve the problem. A practical approach would be to relax these conditions by replacing (Eq. 11*e*) with (Eq. 11*f*):

$$0 \le p_k^T \bar{y}_i \le \sqrt{5}, k = 1,2, \dots ,5;\ i = 1,2, \dots, m. \tag{11f}$$

Thus, DCA methods can be used to solve the relaxed problem [9, 18]. The DCA method is of the primal-dual type and is based on the construction of two strings $\{y^{(k)}\}, \{u^{(k)}\}$ which are calculated at each iteration as follows:

Step 1.$y^{(0)}$ - the initial start approximation, *k*=0.

Step 2. It is determined $u^{(k)} = \nabla \varphi_1(y^{(k)})$.

Step 3. The solution $y^{(k+1)}$ of the convex separable programming problem is established:

$$\left. \begin{array}{c} \varphi_2(y) - \varphi_1(u^{(k)}) - \left(y - y^{(k)}\right)^T u^{(k)} \to min \\ \text{subject to } (11b), (11c), (11d), (11f) \end{array} \right\}.$$

Step 4. If the stop criterion is checked, then STOP. Otherwise, take $k = k + 1$ and move on to Step 2.

6. A solved example

Consider the following example of planning consecutive days off with data:

- 5 days work week;
- Number of workers $m = 6$;
- Number of d_i days off per week for the worker $i = 1, 2, \dots, 6$:

$$d_1 = 2, d_2 = 2, d_3 = 2, d_4 = 3, d_5 = 3, d_6 = 4;$$

- n_k number of workers required on day *k*: $n_1 = 4, n_2 = 3, n_3 = 3, n_4 = 2, n_5 = 2$.

The optimal local solution of the model (Eq. (11a)), (Eq. (11b)), (Eq. (11c)), and (Eq. (11f)) provides the following values for $\bar{y}_i = (y_{i1},\ y_{i2}, y_{i3}, y_{i4}, y_{i5})^T, i = 1, 2, 3, 4, 5, 6$:

$$y_{11} = 0.91423, y_{12} = -0.89614, y_{13} = -0.2341, y_{14} = -0.36601, y_{15} = 0.5002,$$
$$y_{21} = 0.91423, y_{22} = 0.95416, y_{23} = 0.4023, y_{24} = -0.20164, y_{25} = 0.20101,$$
$$y_{31} = 0.91423, y_{32} = 0.89614, y_{33} = -0.2341, y_{34} = -0.36601, y_{35} = 0.5002,$$
$$y_{41} = 1.0459, y_{42} = 1.09108, y_{43} = 0.05124, y_{44} = 0.42181, y_{45} = -0.51623,$$
$$y_{51} = 1.0459, y_{52} = -0.9172, y_{53} = -0.4023, y_{54} = 0.20164, y_{55} = -0.20101,$$
$$y_{61} = 1.5989,\ y_{62} = -0.50201 = y_{63} = y_{64} = \ y_{65}.$$

Considering that $\bar{x}_i = F_0 \bar{y}_i$, we obtain:

$$x_{11} = 5.7632 \cdot 10^{-2}, x_{12} = -8.8091 \cdot 10^{-3}, x_{13} = -1.4474 \cdot 10^{-2}, x_{14} = x_{15} = 1.011,$$
$$x_{21} = 0.99867, x_{22} = 0.95416, x_{23} = -3.6595 \cdot 10^{-2}, x_{24} = 0.10062, x_{25} = 2.7421 \cdot 10^{-2},$$
$$x_{31} = 5.7632 \cdot 10^{-2}, x_{32} = -8.8091 \cdot 10^{-3}, x_{33} = -1.4474 \cdot 10^{-2}, x_{34} = x_{35} = 1.011,$$
$$x_{41} = 1.0459, x_{42} = 0.96225, x_{43} = 0.90525, x_{44} = -0.25499, x_{45} = -0.21019,$$
$$x_{51} = -0.12208, x_{52} = -7.7861 \cdot 10^{-2}, x_{53} = 0.91319, x_{54} = 0.77597, x_{55} = 0.84918,$$
$$x_{61} = -0.18297, x_{62} = 0.93956 = x_{63} = x_{64} = x_{65.}$$

A schedule of consecutive days off generated by this model (rounding x_{ij} to integers) is given in the table 1:

Table 1

A schedule of consecutive days off

Worker	Day					
	1	2	3	4	5	d_i
1	$x_{11} = 0$	$x_{12} = 0$	$x_{13} = 0$	$x_{14} = 1$	$x_{15} = 1$	2
2	$x_{21} = 1$	$x_{22} = 1$	$x_{23} = 0$	$x_{24} = 0$	$x_{25} = 0$	2
3	$x_{31} = 0$	$x_{32} = 0$	$x_{33} = 0$	$x_{34} = 1$	$x_{35} = 1$	2
4	$x_{41} = 1$	$x_{42} = 1$	$x_{43} = 1$	$x_{44} = 0$	$x_{45} = 0$	3
5	$x_{51} = 0$	$x_{52} = 0$	$x_{53} = 1$	$x_{54} = 1$	$x_{55} = 1$	3
6	$x_{61} = 0$	$x_{62} = 1$	$x_{63} = 1$	$x_{64} = 1$	$x_{65} = 1$	4
n_k	4	3	3	2	2	

7. Conclusions

In this paper we considered the problem of planning consecutive days off, formulated as a problem of non-convex quadratic binary programming, which is known to be NP-hard. The structure of the circulating matrix in the objective function allows its diagonalization. The main challenge is to quickly calculate the diagonal matrix Λ_0 and the Fourier matrix F_0, which in our approach reduces the considered problem to a separable programming problem. The characteristic of this transformation is that the matrix F remains well conditioned $(cond(F)) = 1$ regardless of the number of decision variables in the problem formulation. As far as we know, such circulating matrix approaches have not been studied in the literature. The proposed technique allows the use of the DCA algorithm to calculate local suboptimal solutions to the initiated problem, but it is also possible to find the global optimum using, for example, a branch and bound method, or in combination with the classical approximation results of the separable programming problem with a linear programming model.

Conflicts of Interest: The authors declare no conflict of interest.

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