PERFORMANCE MODELING OF COMPUTING PROCESSES USING RECONFIGURABLE DIFFERENTIAL STOCHASTIC PETRI NETS

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INTRODUCTION

A natural modeling framework for many complex systems, such as communication or computer systems and networks, is provided through discrete event systems and, in particular, generalized stochastic Petri nets (*GSPN*) models [3]. However, factors such as huge traffic volumes, increasingly complex operating rules, and performance requirements make such models highly impractical. From an analytical standpoint, traditional models from classical queuing theory fail to capture new features such as complicated traffic source behavior or blocking phenomena. An alternative modeling paradigm for the purpose of analysis and simulation is based on Stochastic Fluid Models (*SFM*). The SFM paradigm allows the aggregation of multiple events into a single event associated with a “significant change” in the system dynamics.

Among the formalisms of *SFM* that are used, the fluid stochastic Petri nets (*FSPN*) [9] and hybrid stochastic Petri nets (*HSPN*) [1, 8] are popular. To make design issues and analysis procedures more transparent with negative-continuous values, we tried to deviate as little as possible from the concepts and the nets of *FSPN* and *HSPN*. Thus, we propose our extension of differential Petri nets (*GDPN*) [5], which we call Generalized Differential Stochastic Petri Net (*GDSPN*), and that is able to represent the behavior of computing processes in a common model. The features of *GDSPN* accept the *negative-continuous place capacity, negative real values for continuous place marking and token-dependent arc cardinalities* that permit to generalize the concept of *GDPN*, *FSPN* and *HSPN*.

1. GENERALIZED DIFFERENTIAL STOCHASTIC PETRI NETS

The problem of state space explosion has challenged numerical solution of Markovian models for a generation. In this paper we propose a means of avoiding this problem for large scale models of repeated components, represented in GSPN. By adopting a continuous approximation of the model behavior we are able to analyze systems of arbitrarily large scale. However, work is progressing on relaxing these assumptions.

Let IN_+ and IR be the sets of discrete natural and real numbers, respectively.

Definition 1. A Generalized differential Petri net (*GDPN*) is a 10-tuple $H\Gamma = <P, T, Pre, Post, Test, Inh, K_p, K_b, G, Pri>$, where:

- P is the finite set of places partitioned into a set of discrete places $P_d = \{p_1, \cdots, p_{n_d}\}$, $n_d = |P_d|$, and a set of continuous places $P_c = \{b_1, \cdots, b_{n_c}\}$, $n_c = |P_c|$, $P = P_d \cup P_c$, $P_d \cap P_c = \varnothing$. The discrete places may contain a natural number of tokens, while the marking of a continuous place is a real number (fluid level). In the graphical representation, a discrete place is drawn as a single circle while a continuous place is drawn with two concentric circles;
- T is a finite set of transitions, that can be partitioned into a set $T_d = \{t_1, \cdots, t_{k_d}\}$, $k_d = |T_d|$ of discrete transitions and a set $T_c = \{u_1, \cdots, u_{k_c}\}$, $k_c = |T_c|$ of continuous transitions, $T = T_d \cup T_c$, $T_d \cap T_c = \varnothing$. A transition $t_j \in T_d$ is drawn as a black bar; a continuous transition $u_l \in T_d$ is drawn as an empty rectangle.
- *Pre*, *Test* and $Inh: P \times T \to Bag(P)$ respectively, are forward flow, test and inhibition functions. $Bag(P)$ are discrete or real-valued multisets functions over *P*. The backward flow function in the multisets of P is $Post: T \times P \to Bag(P)$. These functions define the set of arcs $\mathcal{A}$ and describe the marking-dependent cardinality of arcs connecting transitions with places and vice-versa. Also, the $\mathcal{A}$ set is partitioned into subsets:

$$\mathcal{A} = \mathcal{A}_d \cup \mathcal{A}_c \cup \mathcal{A}_h \cup \mathcal{A}_t \cup \mathcal{A}_s,$$
$$\mathcal{A}_d \cap \mathcal{A}_c \cap \mathcal{A}_h \cap \mathcal{A}_t \cap \mathcal{A}_s = \varnothing.$$

The subset $\mathcal{A}_d$ and $\mathcal{A}_s$ contains respectively the *discrete normal* and *continuous normal* set arcs which can be seen as a function:

$\mathcal{A}_d$: $((P_d \times T_d) \cup (T_d \times P_d)) \times Bag(P) \to IN_+$, and $\mathcal{A}_s$: $((P_c \times T_d) \cup (T_d \times P_c)) \times Bag(P) \to IR$.
The arcs of $\mathcal{A}_d$ and $\mathcal{A}_s$, are drawn as single arrows. The subset of *inhibitory* and *test* arcs is $\mathcal{A}_h$, $\mathcal{A}_t$: $(P \times T) \times Bag(P) \to IN_+$ or that of *continuous inhibitory* and *test* arcs is $\mathcal{A}_h$, $\mathcal{A}_t$: $(P_c \times T) \times Bag(P) \to IR$. These arcs are directed from a place to any kind to a transition of any kind. The *inhibitory* arcs are drawn with a small circle at the end and *test* arcs are drawn as dotted single arrows. It does not consume the content of the source place. The subset $\mathcal{A}_c$ defines the *continuous flow* arcs $\mathcal{A}_c$: $((P_c \times T_c) \cup (T_c \times P_c)) \times Bag(P) \to IR$, and these arcs are drawn as double arrows to suggest a pipe. The arc of a net is drawn if the cardinality is not zero and it is labeled to the arc with a default value being 1;

- $K_p : P_d \to IN_+ \cup \{\infty\}$ is the capacity-function of discrete places and for each $p_i \in P_d$ this is represented by the maximum capacity $K_{p_i}^{\max}$, $0 < K_{p_i}^{\max} < +\infty$, which can contain an natural number of *tokens*. By default, the $K_{p_i}^{\max} \to +\infty$, and it has no blocking effect;
- $K_b : P_C \to IR \cup \{\infty\}$ is the capacity-function of continuous places and for each $b_i \in P_c$ it describes the fluid lower bounds $x_i^{\min}$ and upper bounds $x_i^{\max}$ of the fluid, so that $-\infty < x_i^{\min} < x_i^{\max} < +\infty$. By default, $x_i^{\min} = 0$ and $x_i^{\max} \to +\infty$, and it has no blocking effect;
- $G : T \times Bag(P) \to \{true, false\}$ is the *guard function* defined for each transition. For $t \in T$ a guard function $g(t,\mathcal{M})$ will be evaluated in each marking $\mathcal{M}$, and if it evaluates to *true*, the transition may be enabled, otherwise *t* is disabled (by default it is *true*);
- $Pri: T \times Bag(P) \to IN_+$ defines the priority functions for the firing of each transition. By default it is 0. The enabling of a transition with higher priority disables all the lower priority transitions. ■

The structure of a *GDPN* is static. The dynamics of a net structure is specified by defining its initial marking and its marking evolution rule.

Definition 2. A system stochastic timed marked *GDPN* net (*GDSPN*) is a pair $\mathcal{NH} = <\mathcal{N}, \mathcal{M}_0>$, where $\mathcal{N} = <H\Gamma, \Lambda, W, V>$ is a system timed stochastic timed *GDPN* structure (see Definition 1) with the respective attributes of timed transitions and $\mathcal{M}_0$ is the initial marking of the net so that:

- The set of discrete transitions T_d also is partitioned into two subsets $T_d = T_0 \cup T_\tau$, $T_0 \cap T_\tau = \varnothing$ so that: T_0 is a set of *immediate* discrete transitions and T_τ is a set of *timed* discrete transitions, so that, $\forall t_j \in T_0$ and $\forall t_k \in T_\tau$, $Pri(t_j) > Pri(t_k)$. The immediate transitions are drawn as a black thin bar and timed transitions are drawn as a black rectangle;
- The current marking (state) value of a net depends on the kind of place, and it is described by a pair of vector-columns $\mathcal{M}=(\boldsymbol{m}, \boldsymbol{x})$, where $\boldsymbol{m}$: $P_d \to IN_+$ and $\boldsymbol{x}$: $P_c \to IR$ are the marking functions of respective type of places. The discrete marking $\boldsymbol{m} = (m_i p_i, m_i \geq 0, \forall p_i \in P_d)$ with $m_i p_i$ describe the number $m_i = \boldsymbol{m}(p_i)$ of tokens in discrete place p_i, and it is represented by black dots. The continuous marking $\boldsymbol{x} = (x_k b_k, x_k^{\min} \leq x_k \leq x_k^{\max}, \forall b_k \in P_c)$ with $x_k b_k$ describe the fluid level $x_k = \boldsymbol{x}(b_k)$ in continuous place b_k and it is a real number, also allowed to take *negative* real value. The initial marking of net is $\mathcal{M}_0 = (\boldsymbol{m}_0, \boldsymbol{x}_0)$. Vectors $\boldsymbol{m}_0$ and $\boldsymbol{x}_0$ give the initial marking of discrete places and of continuous places, respectively;
- $\Lambda : T_\tau \times Bag(P) \to IR_+$ is the rate function that maps timed discrete transition onto real nonnegative numbers IR_+. It can be marking dependent. The firing rate $\lambda_j(\mathcal{M})$ define the parameter of negative exponential distribution governing it firing duration for each timed discrete transition of $t_j \in T_\tau$.
- $W : T_0 \times Bag(P) \to IR_+$ is the weight function of immediate discrete transitions $t_k \in T_0$, and this type of transitions is drawn with a black thin bar and has a zero constant firing time.
- $V : T_c \times Bag(P) \to IR$ is the marking dependent fluid rate function of timed continuous transitions $u_j \in T_c$. If u_j is enabled in *tangible* marking $\mathcal{M}$ it fires with rate $V_j(\mathcal{M})$, so that it continuously changes the fluid level of continuous place $b_k \in P_c$. ■

The role of the previous set of arcs and functions will be clarified by providing the enabling and firing rules. Let us denote by m_i the i-th component of the vector $\boldsymbol{m}$, i.e., the number of tokens in discrete place p_i when the marking is $\boldsymbol{m}$, (and x_k denote the k-th component of the vector $\boldsymbol{x}$, i.e. the fluid level in continuous place p_k).

Figure 1 summarizes the graphical representation of all the $\mathcal{NH}$ primitives.

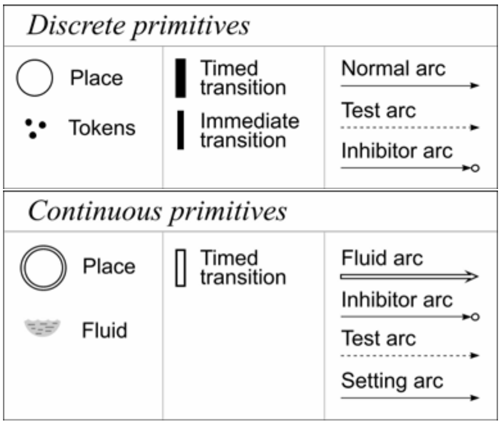

Figure 1. Graphical representation of all the $\mathcal{NH}$ primitives.

Let $T(\mathcal{M})$ be the set of enabled transitions in current marking $\mathcal{M}$. We say that a discrete transition $t_j \in T_d(\mathcal{M})$ is enabled in current marking $\mathcal{M}$ if the following logic expression (enabling condition $ec_d(t_j)$) is verified:

$$ec_d(t_j) = (\underset{\forall p_i \in {}^\bullet t_j}{\wedge} (m_i \geq Pre(p_i, t_j)) \,\&\, (\underset{\forall p_k \in {}^\circ t_j}{\wedge} (m_k < Inh(p_k, t_j)) \,\&\, (\underset{\forall p_l \in {}^* t_j}{\wedge} (m_l \geq Test(p_l, t_j)) \,\&\, (\underset{\forall p_n \in t_j^\bullet}{\wedge} ((K_p - m_n) \geq Post(p_n, t_j)) \,\&\, (\underset{\forall b_i \in {}^\bullet t_j}{\wedge} (x_i \geq Pre(b_i, t_j)) \,\&\, (\underset{\forall b_k \in {}^\circ t_j}{\wedge} (x_k < Inh(b_k, t_j)) \,\&\, (\underset{\forall b_l \in {}^* t_j}{\wedge} (x_l \geq Test(b_l, t_j)) \,\&\, (\underset{\forall b_n \in t_j^\bullet}{\wedge} ((K_b - x_n) \geq Post(x_n, t_j)) \,\&\, g(t_j, \mathcal{M}).$$

The transition $t_j \in T_d(\mathcal{M})$ fires if no other transition $t_k \in T_d(\mathcal{M})$ with higher priority is enabled. If an immediate discrete transition is enabled in current marking $\mathcal{M} = (\boldsymbol{m}, \boldsymbol{x})$, it is *vanishing*. Otherwise, the marking is *tangible* and any timed discrete transition is enabled in it [3, 5]. If several enabled immediate discrete transition $t_j \in T_0(\mathcal{M})$ for $t_j \in {}^\bullet p_i$ are scheduled to fires at the same time in *vanishing* marking $\mathcal{M}$, with the respective weight speeds, $w_j(\mathcal{M})$, the $q_j(\mathcal{M}) = w_j(\mathcal{M}) / \sum_{t_l \in (T_0(M) \& {}^\bullet p_i)} w(t_l, \mathcal{M})$ is the probability that enabled immediate transition $t_j \in T_0$ can *fires*.

Also, we say that a continuous transition $u_j \in T_c(\mathcal{M})$ is enabled and continuously fires in current marking $\mathcal{M}$ if the following logic expression (the enabling condition $ec_c(u_j)$) is verified:

$$ec_c(u_j) = (\underset{\forall b_i \in {}^\bullet u_j}{\wedge} (x_i > 0) \,\&\, (\underset{\forall p_k \in {}^\circ u_j}{\wedge} (m_k < Inh(p_k, u_j)) \,\&\, (\underset{\forall p_l \in {}^* u_j}{\wedge} (m_l \geq Test(p_l, u_j)) \,\&\, (\underset{\forall b_k \in {}^\circ u_j}{\wedge} (x_k < Inh(b_k, u_j)) \,\&\, g(t_j, M) \,\&\, (\underset{\forall b_l \in {}^* u_j}{\wedge} (x_l \geq Test(b_l, u_j)) \,\&\, (\underset{\forall b_n \in u_j^\bullet}{\wedge} ((K_{b_n} - x_n) \geq V_j \cdot Post(x_n, u_j)),$$

and no transition with higher priority is enabled.

An immediate discrete transition t_j enabled in marking $\mathcal{M} = (\boldsymbol{m}, \boldsymbol{x})$ yields a new vanishing marking $\mathcal{M}' = (\boldsymbol{m}', \boldsymbol{x})$. We can write $(\boldsymbol{m}, \boldsymbol{x})\ [t_j>(\boldsymbol{m}', \boldsymbol{x})$. If the marking $M = (\boldsymbol{m}, \boldsymbol{x})$ is tangible, fluid could continuously flow through the flow arcs $\mathcal{A}_c$ of enabled continuous transitions into or out of fluid places. As a consequence, a transition t_c is *enabled* at $\mathcal{M}$ if for every $b \in {}^\bullet u$, $\boldsymbol{x}(b) > 0$, and its *enabling degree* is: $enab(u, \mathcal{M}) = \min_{b \in {}^\bullet u} \{\boldsymbol{x}(b)/Pre(u, b)\}$.

Upon firing, the discrete (continuous) transition removes a specified number (quantity) of tokens (fluid) for each discrete (fluid) input place, and deposits a specified number (quantity) of tokens (fluid) for each discrete (fluid) output place. The levels of fluid places can change the enabling/disabling of transitions.

We allow the firing rates and the enabling functions of the timed discrete transitions, the firing speeds and enabling functions of the timed continuous transitions, and arc cardinalities to be dependent on the current state of the $\mathcal{NH}$, as defined by the current marking $\mathcal{M}$.

4. DYNAMIC REWRITING GDSPN

In this section we introduce the model of *descriptive dynamic net rewriting systems.*

Let $X \rho Y$ be a binary relation. The *domain* of ρ is the $Dom(\rho) = \rho Y$ and the *codomain* of ρ is the $Cod(\rho) = X\rho$. Also, let $A = <Pre, Post, Test, Inh>$ be a set of arcs belonging to net $\mathcal{NH}$ $\mathcal{N} = <H\Gamma$,

Λ, W, V >, $H\Gamma$ = < P, T, Pre, $Post$, $Test$, Inh, K_p, K_b, G, Pri > (see *Definition* 2).

Definition 3**:** A *dynamic rewriting GDPN* is a system RH =< $\mathcal{N}$, R, ϕ, G_r, M >, where:

- $\mathcal{N}$= < $H\Gamma$, Λ, W, V > and $R = \{r_1, ..., r_k\}$ is a finite set of discrete rewriting rules (*DR*) about the run-time structural modification of a net, so that $P \cap T \cap R = \varnothing$. In the graphical representation, the *DR* rule is drawn as two embedded empty rectangles;
- $\phi : E \to \{T_D, R\}$ is a function which indicates for every rewriting rule the type of event that can occur and $E = T_D \cup R$ denote the set of *events* of the net;
- $G : E \times Bag(P) \to \{true, false\}$ is the *event rule guard function* associated with $e \in E$, and $G_r : R \times Bag(P) \to \{true, false\}$ is the *rewriting rule guard function* defined for each rule of $r \in R$, respectively. For $\forall e \in E$, the function $g_e(M) \in G$ and $g_r(M) \in G_r$ will be evaluated in each marking and if they are evaluated to *true*, the event *e* may be *enabled*, otherwise it is *disabled*. The default value of $g_e(M) \in G$ and $g_r(M) \in G_r$ is *true* in current marking M.

Let RN=<RH, M> be represented by the descriptive expression $DE_{R\Gamma}$ and DE_{RN}, respectively [8]. A dynamic rewriting structure modifying rule $r \in R$ of *RN* is a map $r : DE_L \triangleright DE_W$, where the *codomain* of the $\triangleright$ *rewriting operator* is a fixed descriptive expression DE_L of a subnet RN_L of current net *RN*, where $RN_L \subseteq RN$ with $P_L \subseteq P$, $E_L \subseteq E$ and the set of arcs $A_L \subseteq A$, and the *domain* of the $\triangleright$ is a descriptive expression DE_W of a new RN_W subnet with $P_W \subseteq P$, $E_W \subseteq E$ and set of arcs A_W. The rewriting operator $\triangleright$ represents the binary operation which produces a *structure change* in the DE_{RN} and the net *RN* by replacing (rewriting) the fixed current DE_L of the subnet RN_L (DE_L and RN_L are dissolved) with the new DE_W of the subnet RN_W, now belonging to the new modified resulting $DE_{RN'}$ of the net $RN' = (RN \setminus RN_L) \cup RN_W$ with $P' = (P \setminus P_L) \cup P_W$, $E' = (E \setminus E_L) \cup E_W$, and the set of $A' = (A \setminus A_L) \cup A_W$, where the meaning of \ (and $\cup$) is operation of removing (adding) RN_L from (RN_W to) the net *RN*. In this new net RN', obtained by execution (firing) of enabled rewriting rule $r \in R$, the places and events with the same attributes which belong to RN' are fused. By default, the rewriting rules $r : DE_L \triangleright \varnothing$ or $r : \varnothing \triangleright DE_W$ describe the rewriting rule holding the $RN' = (RN \setminus RN_L)$ or $RN' = (RN \cup RN_W)$.

A state configuration of a net *RN* is a pair ($R\Gamma, s$), where $R\Gamma$ is the current structure of net RH together with a current state $s = (M, \beta(M))$. The ($R\Gamma_0, s_0$) with $P_0 \subseteq P, E_0 \subseteq E$ and state s_0 is called the initial state configuration of a net *RN*. ■

Figure 2 summarizes the graphical representation of *RH* discrete rewriting primitives.

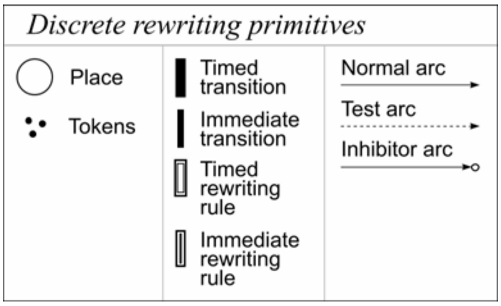

Figure 2. Discrete rewriting primitives of RH.

Enabling and Firing of Events. The enabling of events depends on the of the event e_j is enabled in current marking *M* if marking of all places. We say that a transition $t_j \in T_D$ the enabling condition $ec_d(t_j, M)$ is described in [7] and is verified.

The discrete rewriting rule $r_j \in R$, that changes the structure of *RN*, is enabled in current marking *M* if the $ec_d(e_j)$ and the $g_e(r_j, M))$ are verified.

Let $T_D(M)$ and $R(M)$, $T_D(M) \cap R(M) = \varnothing$, be the sets of enabled discrete transitions and enabled rewriting rule in current marking *M*, respectively. We denote the set of enabled events in a current marking M by $E(M) = T_D(M) \cup R(M)$.

The event $e_j \in E(M)$ fires if no other event $e_k \in E(M)$ with higher priority is enabled. Hence, for each e_j *if* $((\phi_j = t_j) \vee (\phi_j = r_j) \wedge (g_e(e_j, M) = False))$ *then* the firing of transition $t_j \in T_D(M)$ or rewriting rule $r_j \in R(M)$ changes only the current marking: $(R\Gamma, s) \xrightarrow{e_j} (R\Gamma, s') \Leftrightarrow (R\Gamma = R\Gamma$ and in $R\Gamma$ the $M[e_j > M')$. Also, for e_j event ***if*** $((\phi_j = r_j) \wedge (g_r(r_j, M) = True))$ ***then*** the event e_j occurs to firing of rewriting rule r_j and it changes the configuration and marking of the current net in the following way:

$(R\Gamma, s) \xrightarrow{r_j} (R\Gamma', s')$, $M[r_j > M'$.

The accessible state graph of a $RN = < R\Gamma, M >$ net is the labeled directed graph whose nodes are the states and whose arcs which are labeled with events or rewriting rules of RN are of two kinds:

a) *firing* of an enabled $e_j \in E(M)$ event determines an arc from the state $(R\Gamma, s)$ to the state $(R\Gamma, s')$ which is labeled with event e_j when this event can fire in the net configuration $R\Gamma$ at marking M and leads to a new state:

$$s' : (R\Gamma, s) \xrightarrow{e_j} (R\Gamma', s') \Leftrightarrow$$

$$(R\Gamma = R\Gamma' \text{ and } M[e_j > M' \text{ in } R\Gamma);$$

b) *change configuration*: arcs from state $(R\Gamma, s)$ to state $(R\Gamma', s')$ labelled with the rewriting rule $r_j \in R$, so that $r_j : (R\Gamma_L, M_L) \triangleright (R\Gamma_W, M_W)$ which represent the change configuration of current RN net: $(R\Gamma, s) \xrightarrow{r_j} (R\Gamma', s')$ with $M[r_j > M'$.

3. ANALYTICAL DESCRIPTION OF SGDPN MODELS

For the analysis of an *SGDPN* the underlying stochastic process must be defined. The node of the discrete part reachability graph consists of all discrete markings supplemented by vector of random variable for the fluid levels. It gives rise to a stochastic process, which is a Markov process in continuous time with mixed state space [4, 6]. The bounded, live and reversible *GSDPN* are isomorphic to continuous-time hybrid Markov chains (*CHMC*) due to the memory less property of exponential distribution.

We denote the set of all *markings* (or the partially discrete and partially continuous state space) of the net by $S = IN_+^{|P_d|} \times IR^{|P_c|}$. In the following we denote by S_d and S_c the discrete and the continuous component of the state space, respectively, so that $S = S_d \cup S_c$, $S_d \cap S_c = \varnothing$.

The current marking $\mathcal{M} = (\boldsymbol{m}, \boldsymbol{x})$ of $\mathcal{NH}$ evolves in time. We denote the time by τ, and $\mathcal{M}(\tau)$ the current marking at time τ of the marking process $S(\tau) = \{\, \boldsymbol{m}(\tau),\ \boldsymbol{x}(\tau),\ \tau > 0 \,\}$ of $\mathcal{NH}$ net.

In the $\mathcal{NH}$, the instantaneous fluid speed (*dynamic balance*) $\upsilon_{i,k}(\mathcal{M})$ that change of fluid level in continuous place $b_i \in P_c$ in current marking $\mathcal{M} = (\boldsymbol{m}_k, \boldsymbol{x})$, $\boldsymbol{m}_k \in S_d$, $\boldsymbol{x} \in S_c$ is given by: $\upsilon_{i,k}(\mathcal{M}) = \upsilon_{i,k}^{+}(\mathcal{M}) - \upsilon_{i,k}^{-}(\mathcal{M})$, $i = \overline{1, n_c}$, $n_c = |P_c|$, where for any given $u_k, u_n \in T_c(\mathcal{M})$, the $\upsilon_i^{+}(\mathcal{M})$ is an input instantaneous fluid speed of continuous place $b_i \in P_c$ and $\upsilon_i^{-}(\mathcal{M})$ is an output instantaneous fluid speed of this place:

$$\upsilon_{i,k}^{+}(\mathcal{M}) = \sum\nolimits_{u_k \in {}^{\bullet}b_i} [\, V_k(\mathcal{M}) \cdot Pre(u_k, b_i) \,],$$

$$\upsilon_{i,k}^{-}(\mathcal{M}) = \sum\nolimits_{u_n \in b_i^{\bullet}} [\, V_n(\mathcal{M}) \cdot Post(u_n, b_i) \,].$$

Live and bounded *HSPN* are isomorphic to continuous-time hybrid Markov chain (*CHMC*) due to the memory less properly of exponential distribution [3].

Let S_d be the discrete set of state space of *CHMC* and let $\boldsymbol{D}(\ x) = [d_{l,j}(x)]$, $i, j = 0, \dots, |S_d|$ be the *dynamic matrix* of transition rates derived from the rate function of discrete transitions of discrete part *GDSPN* [1, 8].

The dynamic balances $\upsilon_i(\mathcal{M})$ that changes levels for each continuous place b_i in discrete marking $\boldsymbol{m}_k \in S_d$ are collected in the diagonal matrice:

$$\upsilon_k(\boldsymbol{x}) = diag(\upsilon_{i,0}(\boldsymbol{x}), \dots,\ \upsilon_{i,k}(\boldsymbol{x})),$$

$$i = 1, \dots, n_c = |P_c|.$$

The 3-tuple $(\boldsymbol{m}(\tau), \boldsymbol{x}(\tau); \upsilon_k(\boldsymbol{x}))$ describes the state of *CHMC* chain. The transient probability of being in discrete state $\boldsymbol{m}_k$ with fluid levels in an infinitesimal environment around x_i, for all continuous places $b_i \in P_c$ are called the fluid density probability and are denoted by $f_k(\boldsymbol{x}, \tau)$. Let $x_i^{\min} = {}^{-}h_i$ and $x_i^{\max} = {}^{+}h_i$. Let also $\rho_k^{-}(\boldsymbol{x}, \tau)$ or $\rho_k^{+}(\boldsymbol{x}, \tau)$ be a probability mass if $\boldsymbol{x}(\tau)$ has at least one component equal to $x_i = {}^{-}h_i$ or $x_i = {}^{+}h_i$, respectively.

Using the approach described in [8, 9] we have derived the Chapman-Kolmogorov forward equations that are in the following:

- for internal fluid levels values $\forall\ {}^{-}h_i < x_i < {}^{+}h$ of $b_i \in P_c$:

$$\frac{\partial}{\partial \tau} f_k(\boldsymbol{x}, \tau) + \sum_{i=1}^{n_c} \frac{\partial}{\partial x_i} (f_k(\boldsymbol{x}, \tau) \cdot \upsilon_{i,k}(\boldsymbol{x})) = \sum_{l=0}^{|S_d|} (d_{l,k}(\boldsymbol{x}, \tau) \cdot f_l(\boldsymbol{x}, \tau)), \qquad (1)$$

$$k = 0, \dots, N_d,\ N_d = |S_d|.$$

For the boundary conditions two different cases arise, depending on the direction of the fluid flow:

- for the lower boundary fluid levels values $x_i = {}^{-}h_i$ of $b_i \in P_c$:

$\rho_k^-(\boldsymbol{x},\tau)=0$ if $(x_i = {}^{-}h_i) \wedge (\cdot\upsilon_{i,k}(\boldsymbol{x})>0)$, and

$$\frac{\partial}{\partial \tau}\rho_k^-(\boldsymbol{x},\tau)+\sum_{\forall i: x_i = {}^{-}h_i}(f_k(\boldsymbol{x},\tau)\cdot|\upsilon_{i,k}(\boldsymbol{x})|) + \sum_{\forall i: x_i > {}^{-}h_i}(\frac{\partial}{\partial x_i} f_k(\boldsymbol{x},\tau)\cdot\upsilon_{i,k}(\boldsymbol{x})) = \sum_{l=0}^{|S_d|}(d_{l,k}(\boldsymbol{x},\tau)\cdot\rho_l^-(\boldsymbol{x},\tau)),\text{ if } (x_i = {}^{-}h_i)\wedge$$

$(\upsilon_{i,k}(\boldsymbol{x})<0)$, $\forall b_i \in P_c$, $\forall\ m_k \in S_d$; (2)

- for the upper boundary fluid levels values $x_i = {}^{+}h_i$ of $b_i \in P_c$: $\rho_k^+(\boldsymbol{x},\tau)=0$ if $(x_i = {}^{+}h_i)\wedge(\cdot\upsilon_{i,k}(\boldsymbol{x})<0)$, and (3)

$$\frac{\partial}{\partial \tau}\rho_k^-(\boldsymbol{x},\tau)+\sum_{\forall i: x_i = {}^{+}h_i}(f_k(\boldsymbol{x},\tau)\cdot\upsilon_{i,k}(\boldsymbol{x})) + \sum_{\forall k:\ {}^{-}h_i<x_i<{}^{+}h_i}\frac{\partial}{\partial x_i}(f_k(\boldsymbol{x},\tau)\cdot\upsilon_{i,k}(\boldsymbol{x},\tau)) = \sum_{l=0}^{|S_d|}(d_{l,k}(\boldsymbol{x},\tau)\cdot\rho_l^+(\boldsymbol{x},\tau)),\quad\text{if}$$

$(x_i = {}^{+}h_i)\wedge(\cdot\upsilon_{i,k}(\boldsymbol{x})>0)$, $\forall b_i \in P_c$, $\forall\ m_k \in S_d$

Assuming the system converges to a stationary solution of equations (1), (2) and (3), the stationary fluid density function and fluid mass function exists $f_k(\boldsymbol{x})=\lim_{\tau\to\infty} f_k(\boldsymbol{x},\tau)$, $\rho_k^-(\boldsymbol{x})=\lim_{\tau\to\infty}\rho_k^-(\boldsymbol{x},\tau)$ and $\rho_k^+(\boldsymbol{x})=\lim_{\tau\to\infty}\rho_k^+(\boldsymbol{x},\tau)$ only if the system is stable. Stability conditions of *GDSPN* are still a research topic.

For these equation systems the steady-state distribution exists when the underlying of *GDSPN* discrete part is bounded, life, reinitialized and the following relations are verified:

$$\lim_{\forall h_i^+\to\infty}\sum_{\forall \boldsymbol{m}_k\in S_d}(\pi_k(\boldsymbol{x})\cdot\upsilon_{i,k}(\boldsymbol{x}))<0,\ \forall b_i \in P_c, \qquad (4)$$

where $\pi_k(\boldsymbol{x})$ is the stationary probability of discrete marking $m_k \in S_d$ determined by the underlying continuous-time Markov chain (*CTMC*) of *GDSPN* discrete part [4, 10]. These relations are obtained by solving the following linear system equations that describe the behavior of *CTMC*:

$$\vec{\pi}(\boldsymbol{x})\cdot\boldsymbol{D}(\boldsymbol{x})=\boldsymbol{0},\quad \sum_{\forall \boldsymbol{m}_k\in S_d}\pi_k(\boldsymbol{x})=1. \qquad (5)$$

Over the ${}^{-}h_i < x_i < {}^{+}h$ internal fluid levels value intervals the stationary distribution of $f_k(\boldsymbol{x})$, $k=0,\dots,N_d$, $N_d = |S_d|$, satisfies:

$$\sum_{i=1}^{n_c}\frac{\partial}{\partial x_i}(f_k(\boldsymbol{x})\cdot\upsilon_{i,k}(\boldsymbol{x}))=\sum_{l=0}^{|S_d|}(d_{l,k}(\boldsymbol{x})\cdot f_l(\boldsymbol{x})),\quad \forall\ {}^{-}h_i < x_i < {}^{+}h. \qquad (6)$$

For the boundary conditions, depending on the direction of the fluid flow [3, 6]:

- for the lower boundary fluid levels values $x_i = {}^{-}h_i$ of $b_i \in P_c$:

$\rho_k^-(\boldsymbol{x})=0$ if $(x_i = {}^{-}h_i)\wedge(\cdot\upsilon_{i,k}(\boldsymbol{x})>0)$, and

$$\sum_{\forall i: x_i = {}^{-}h_i}(f_k(\boldsymbol{x},\tau)\cdot|\upsilon_{i,k}(\boldsymbol{x})|) + \sum_{\forall k: x_i > {}^{-}h_i}(\frac{\partial}{\partial x_i} f_k(\boldsymbol{x}) \cdot\upsilon_{i,k}(\boldsymbol{x}|) = \sum_{l=0}^{|S_d|}(d_{l,k}(\boldsymbol{x})\cdot\rho_l^-(\boldsymbol{x})), \qquad (7)$$

if $(x_i = {}^{-}h_i)\wedge(\upsilon_k(\boldsymbol{x})<0)$, $\forall b_i \in P_c$, $\forall\ m_k \in S_d$;

- for the upper boundary fluid levels values $x_i = {}^{+}h_i$ of $b_i \in P_c$:

$\rho_k^+(\boldsymbol{x})=0$ if $(x_i = {}^{+}h_i)\wedge(\upsilon_{i,k}(\boldsymbol{x})<0)$, and

$$\sum_{\forall i: x_i = {}^{+}h_i}(f_k(\boldsymbol{x},\tau)\cdot\upsilon_{i,k}(\boldsymbol{x})) + \sum_{\forall i: x_i > {}^{+}h_i}(\frac{\partial}{\partial x_i} f_k(\boldsymbol{x})\cdot\upsilon_{i,k}(\boldsymbol{x})) + \sum_{i:\ {}^{-}h_i<x_i<{}^{+}h_i}\frac{\partial}{\partial x_i}(f_k(\boldsymbol{x})\cdot\upsilon_{i,k}(\boldsymbol{x})) = \sum_{l=0}^{|S_d|}(d_{l,k}(\boldsymbol{x})\cdot\rho_l^+(\boldsymbol{x})), \qquad (8)$$

if $(x_i = {}^{+}h_i)\wedge(\cdot\upsilon_{i,k}(\boldsymbol{x})>0)$, $\forall b_i \in P_c$, $\forall\ m_k \in S_d$

The *GDSPN* model solution problem is in general not analytically tractable. The numerical solution algorithms proposed in [2, 6, 9] are applicable only when the interactions between the discrete and continuous portions of the net satisfy fairly strong assumptions.

To obtain the steady-state solution of the dynamics for the stationary fluid mass probability:

$$\rho_k(\boldsymbol{x}) = (\rho_k^-(\boldsymbol{x}) + \rho_k^+(\boldsymbol{x}) + \int_{{}^{-}\vec{h}}^{\vec{x}} f_k(\boldsymbol{x})\boldsymbol{dx},$$

$\forall m_k \in S_d$

with $\int_{{}^{-}\vec{h}}^{\vec{x}}\boldsymbol{dx} = \int_{{}^{-}h_{n_c}}^{x_{n_c}}\dots\int_{{}^{-}h_1}^{x_1} dx_1 \dots dx_{n_c}$

of the *GDSPN* model has been computed by using an extension of the finite difference solution technique proposed in [6, 9], which confirms to the boundary conditions and satisfies the normalization condition at the same time. These are quite complex and hard to solve.

Hence, discrete-event simulation becomes an important alternative avenue to study the behavior and the solution of *GDSPNs* models under some restrictions. However, due to the mixed nature of the state space, with discrete and continuous components and arbitrary interactions between them, simulation also poses several challenges that we address. In [3] are characterized the types of these interactions as belonging to one of the several restricted classes of models and are proposed a better suited, and faster, simulation algorithms can be employed for the solution and to predict the behavior.

Thus, for visual simulation and analysis of *GDSPNs* we have elaborated the VHPNtool [7].

Continuous performance measures can be classified as *fluid state measures* and *flow measures*. Fluid state measures give the probability of a condition connected to the fluid levels in the net, while flow measures can be considered as the continuous counterpart of discrete throughput measures.

In order to show the applicability of *GDSPNs* we consider a pipe-line hybrid computing system consisting of three processing elements PE_j, j=1,2,3 (see figure 3). Each element PE_j can be in two local states $\alpha_j \in \{0, 1\}$. In the active state $\alpha_j = 1$, the element PE_j with speed V_j will, in continuous mode, decrease the level x_k of buffer b_k , k = 4-j and in the same time it will in continuous mode increase the level x_j of buffer b_j, j=1, 2, 3. In the passive state $\alpha_j = 0$ it will not change them anymore. The time sojourn of each element PE_j in the states $\alpha_j = 1$ or $\alpha_j = 0$ are negative exponentially distributed random variables with rates λ_j or μ_j.

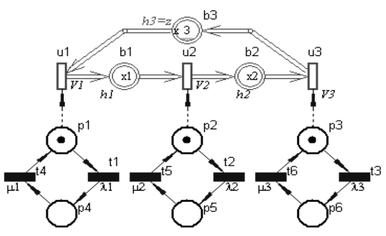

Figure 3. Translation of DE_{sys} in NH_{sys}.

The blocking effect of PE_j in $\alpha_j = 1$ is represented by capacity $K_{b_j} = h_j$ of buffer b_j if this is full. Further, we will note x_1=x, x_2=y and x_3=z. The net NH_{sys} has four P-invariants that cover all places: $m(p_j) + m(p_{j+3})$=1, j=1,2,3 for discrete places and $x + y + z = h$ for continuous places. For the initial marking $m(p_j)$=1, x_0=y_0=0, z_0= h_3=h_1+h_2, and the current state of *NH1* can be described by 7-tuple $(\alpha_1\alpha_2\alpha_3, xy; \beta_x, \beta_y)$, where β_x and β_y are respectively dynamic balances of buffers b_1 and b_2.

The analytical analysis of underlying hybrid continuous time Markov Chain *HMC* of this NH_{sys} model in general case is very difficult. For this analysis is necessary to use the special tool.

Here we give a simplified case for $\lambda_2 = \lambda_3 = 0$, where the elements PE_2 and PE_3 always will be in active state $\alpha_2 = \alpha_3 = 1$ and in this way, the element PE_2 (respective PE_3) with the speed V_2 (respective V_3), will transfer the content of buffer b_1 (respective b_2) in buffer b_2 (respective b_3).

The behavior of NH_{sys} depends on the ratio between speeds V_j. For $V_1 > V_2 > V_3$ the chain *HMC1*, with the respective internal and boundary states, in considerate case, is represented in figure 5, where the discrete marking is $m_i \in \{0, 1\}$ because the element PE_i can be or in passive or in active state.

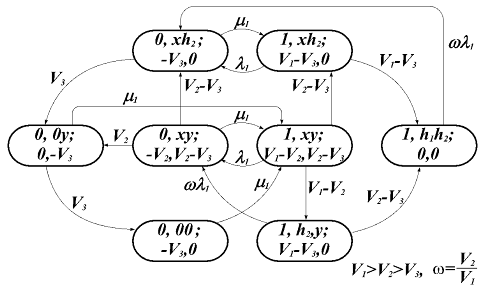

Figure 5. *HMC1* of NH_{sys}.

Let $f_i(x, y)$ denote the steady-state fluid density of *CHMC1* in current marking (m_i, xy), i=0,1. For each internal state (m_i, xy; v_x, v_y), 0<x<h_1 and 0<y<h_2 of the chain *CHMC1* the $f_i(x, y)$ obeys the following system of partial differential equations (*PDE*):

$$-V_2 \cdot \frac{\partial f_0(x,y)}{\partial x} + (V_2 - V_3) \cdot \frac{\partial f_0(x,y)}{\partial y} +$$
$$\mu_1 f_0(x,y) = \lambda_1 f_0(x,y); \qquad (9)$$
$$(V_1 - V_2) \cdot \frac{\partial f_1(x,y)}{\partial x} + (V_2 - V_3) \cdot \frac{\partial f_1(x,y)}{\partial y} +$$
$$\lambda_1 f_1(x,y) = \mu_1 f_0(x,y) \cdot$$

To write the boundary equation directly from graph of chain *HMC1* we introduce the notation: $\pi_i(0)$ or $\pi_i(h_1)$, which are the probabilities of boundary states of buffer b_1 for x=0 or x=h_1, but Q(0) or Q(h_2) of buffer b_2 for y=0 or y=h_2, respectively.

For each state with $\omega = V_2/V_1$ we can write the steady-state probabilities from the boundary equations: $\omega\lambda_1 \cdot \pi_1(h_1) = V_3 \cdot \varphi_0(h_1)$;

$$\omega\lambda_1 \cdot \pi_1(h_1) = (V_2 - V_3) \cdot \varphi_1(h_1);$$

$$\omega\lambda_1 \cdot Q_1(h_2) = (V_2 - V_3) \cdot \varphi_1(h_2);$$
$$\mu_1 \cdot \pi_0(0) = (V_1 - V_2) \cdot \varphi_1(0);$$
$$\mu_1 \cdot Q_0(0) = V_3 \cdot \psi_1(0);$$
$$\mu_1 \cdot Q_0(0) \cdot \psi_0(h_2) = V_3 \cdot \varphi_0(0) Q_0(h_2).$$

Solutions of these equation systems permit us to determine the distribution of steady-state probability and the performance indicators of system, i.e. the average levels $\hat{x}$ and $\hat{y}$ in buffers b_1 and b_2 are:

$$\hat{x} = C \cdot [(V_3 h_1/\mu_1\lambda_1 + (1 + a_1)(\gamma_1 h_1 - 1)/b_1 + + (1 + a_1)/\gamma_1^2,\ b_1 = \gamma_1^2 e^{\gamma_1 h_1}$$

$$\hat{y} = C \cdot [(h_2((V_1 - V_2)/(V_2 - V_3) + (V_2 - V_3)a_2/\mu\lambda_1) + D],$$

where: $D = (1 + a_2)(\gamma_2 h_2 - 1)/\gamma_2^2)e^{\gamma_2 h_2} + (1 + a_2)/\gamma_2^2$,

$$a_2 = A/a_1,\ \gamma_1 = \delta_1 \mu_1/V_2 - \lambda_1/(V_1 - V_2),$$
$$\delta_1 = \mu_1/V_3 - \lambda_1/(V_1 - V_3),\ a_1 = V_3/(V_2 - V_3),$$
$$\gamma_2 = (A\mu_1 + \gamma_1 V_2 - \lambda_1)/(V_2 - V_3).$$

The value of C is a constant obtained from the normalization condition, but γ_2 and A are obtained like solution of following character-ristic equation of system PDE :

$A^2 + b \cdot A - \rho = 0$, where $\rho = \lambda_1/\mu_1$, and $b = 1 + V_1/V_2 - (\rho(2V_1 - V_2)/(V_1 - V_2))$.

From this characteristic equation and from the normalization condition, that density probability always is a positive value, we obtain the solution, A>0.

The time redundancy is: $\hat{\tau}_x = \hat{x}/V_2$ and $\hat{\tau}_y = \hat{y}/V_3$. The same considerations hold for system throughput.

4. CONCLUSIONS

In this paper we propose the generalized differential stochastic Petri nets (GDSPN) for performance modeling of discrete-continuous computing processes. The features of GDSPN accept the negative-continuous place capacity, negative real values for continuous place marking and marked-dependent arc cardinalities. With our approach, the modeling power of fluid models is extended to include the case with fluid-dependent rates. Also, we provide the set of partial differential equations and boundary conditions that determines the stationary behavior and we discuss potential numerical methods that evaluate the stationary distribution based on this description.

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Recommended for publication:28.08.2013.