Stability Conditions of Linear Discrete-Time Systems by Characteristic Equation Coefficients

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Abstract. A linear discrete-time system (LDTS) is deemed stable if the poles of its transfer function are located within the unit circle in the complex z-plane. To verify pole locations, techniques such as the Schur-Cohn criterion or the Jury test can be employed. Alternatively, an indirect approach involves mapping the interior of the unit circle to the left half-plane of the complex *w*-plane using the bilinear transformation, which allows for the application of stability criteria designed for continuous-time systems. However, both direct and indirect methods become increasingly complex as the system's order increases, complicating the synthesis process. Given that modern automatic control systems, especially those involving dynamic objects, are often high-dimensional, there is a significant need for simplified stability criteria to facilitate the synthesis process.

In this work, simplified stability conditions for LTDS have been identified, conditions expressed as relationships among the coefficients of the characteristic equation in the *z*-plane, *w*-plane, or *ζ*-plane:

1. For a LDTS with the characteristic equation

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_2z^2 + a_1z + a_0 = 0,$$

a necessary condition for stability is that the inequality $|ai| < C(n, i)$ holds, where $C(n, i)$ are the binomial coefficients (i = 0, 1, … n-1).

2. For a LDTS with the characteristic equation in the *ζ*-plane [1]

$$\zeta^n + b_{n-1}\zeta^{n-1} + b_{n-2}\zeta^{n-2} + \dots + b_2\zeta^2 + b_1\zeta + b_0 = 0,$$

the inequality $\lambda_i < b_{i-1}b_{i+2}/(b_ib_{i+1}) < 1$, holds ($i$ = 1, 2, …, n-2),.

If any of these inequalities are violated, the system is unstable.

For example, consider the LDTS with the characteristic equation

$$z^3 + z^2 + z + 2 = 0$$

The bilinear transformation, the Stodola criterion, and Routh's table, show that the system is unstable [2]. Applying Condition 1, the same conclusion can be drawn without further calculation, because $a_0 = 2 > C(3,3) = 1$.

Transitioning from the z-plane to the w-plane using the bilinear transformation $z = (w + 1)/(w – 1)$ enables the extension of methods proposed in [3] for the synthesis of discrete systems, as follows:

3. For the stability of a LDTS with the characteristic equation

$w^n + c_{n-1}w^{n-1} + c_{n-2}w^{n-2} + ... + c_2w^2 + c_1w + c_0 = 0$,

it is sufficient to satisfy one of the following conditions:

$$\lambda_i < c_{i-1}c_{i+2}/(c_ic_{i+1}) < 0{,}465 \qquad i = 1, 2, ..., n\text{-}2; \qquad (1)$$

$$\lambda_i + \lambda_{i+1} < 0{,}89 \qquad i = 1, 2, ..., n\text{-}3; \qquad (2)$$

$$\lambda_i + \lambda_{i+1} + \lambda_{i+2} < 1 \qquad i = 1, 2, ..., n\text{-}4. \qquad (3)$$

Satisfying any of these conditions guarantees the stability of the system. In this case the synthesis of a stable LDTS involves the following steps:

1. Applying the bilinear transformation $z = (w + 1)/(w – 1)$ to obtain the characteristic equation in the w-plane.
2. Using 3 to calculate the coefficients of the characteristic equation.
3. Transforming back to the z-plane to complete the determination of the stable system's characteristic equation coefficients.

The proposed stability conditions not only simplify the evaluation of LDTS stability, but, more importantly, significantly facilitate the synthesis of high-dimensional discrete control systems.

References

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