FINITE ELEMENTS IN THE ANALYSIS OF OPEN THIN-WALLED BARS SUBJECTED TO TORSION

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INTRODUCTION

A lot of structural members are made of thin walled open, closed or mixed bars. Even component substructures of a building can be considered thin-walled bars, as the central cores in the high-rise buildings. Their analysis involves a lot of complex problems than can be more easily and accurately solved by using the finite element method (MEF).

In this method, the thin-walled bars are divided into finite elements, their state of loading being in the most cases combined bending and torsion.

Their effects can be individually considered and than superposed.

In the present paper there are presented two types of finite elements used in the analysis of thin-walled open bars considered to be subjected only to torsion that is in fact a restrained torsion, so that both effects of St. Venant torsion and warping must be determined.

2. FINITE ELEMENT DERIVED BY USING THIRD DEGREE SHAPE FUNCTIONS

Generally, the substructures or structural elements modelled as thin walled open bars are acted by distributed twisting moments that occur due to the eccentricities of forces with respect to the shear centres and also by concentrated twisting moments and bi-moments.

The stress and strain state is evaluated by using the finite element method in the variant of Galerkin's procedure, where, as shape functions (l'Hérmite functions) are adopted polynomials of different degrees. Generally, the polynomials of higher degree assure more accurate results.

It is considered a finite element (fig. 1) with two degrees of freedom at each node: the twisting angle $\varphi_i^{/}$ and its first order derivative, $\varphi_i^{/}$, that is the warping.

The corresponding nodal forces are represented by the total twisting moment M_x and

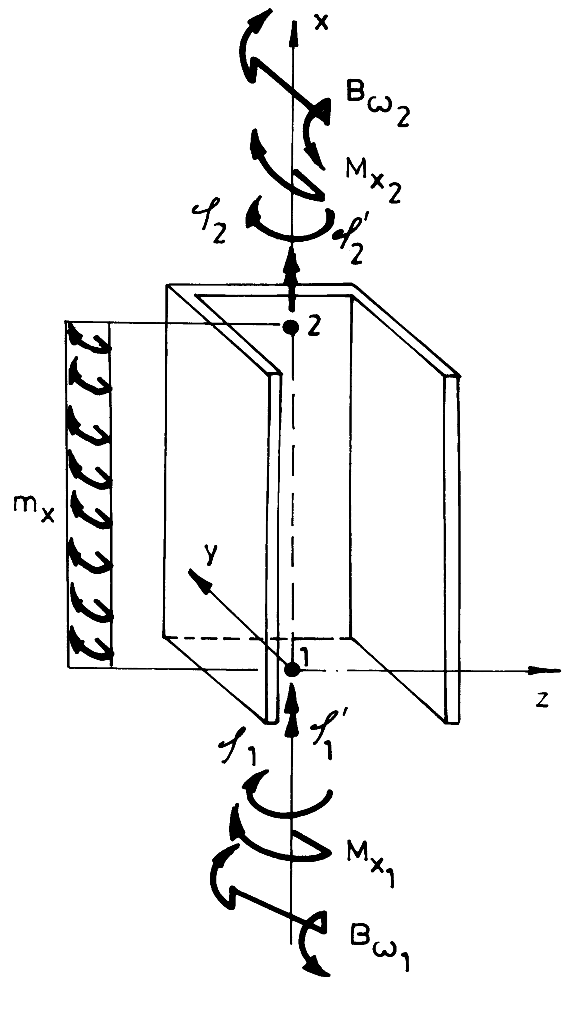

Figure 1.

by the bi-moment B_ω.

There are expressed the column vector of nodal displacements and the column vector of nodal forces:

$$d_e = \begin{Bmatrix} \varphi_1 \\ \varphi_1' \\ \varphi_2 \\ \varphi_2' \end{Bmatrix} \qquad S_e = \begin{Bmatrix} M_{x_1} \\ B_{\omega_1} \\ M_{x_2} \\ B_{\omega_2} \end{Bmatrix} \tag{1}$$

The twisting angle $\varphi_e(x)$ which must satisfy the torsion governing equation

$$EI_\omega \cdot \frac{d^4\varphi}{dx^4} - GI_t \cdot \frac{d^2\varphi}{dx^2} = -m_x$$

for a thin walled open member is approximated by a third degree polynomial:

$$\varphi_e(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \tag{2}$$

or

$$\varphi_e(x) = P^T \cdot \alpha \tag{3}$$

where

$$P^T = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}$$
$$\alpha = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T \tag{4}$$

By using the boundary conditions:

$$\begin{aligned} \varphi(0) &= \varphi_1; & \varphi(\ell) &= \varphi_2 \\ \varphi'(0) &= \varphi_1'; & \varphi'(\ell) &= \varphi_2' \end{aligned} \tag{5}$$

the vector of generalised co-ordinates $\{\alpha\}$ can be expressed in terms of nodal displacements:

$$\alpha = A \cdot d_e \tag{6}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{\ell^2} & -\frac{2}{\ell} & \frac{3}{\ell^2} & -\frac{1}{\ell} \\ \frac{2}{\ell^3} & \frac{1}{\ell^2} & -\frac{2}{\ell^3} & \frac{1}{\ell^2} \end{bmatrix} \tag{7}$$

By substituting (6) in (3), the expression that states the relation between the displacement field and the nodal displacements is obtained:

$$\varphi_e(x) = N^T \cdot d_e \tag{8}$$

where N represents the shape function matrix:

$$N^T = P^T \cdot A = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \tag{9}$$

$$N_1 = 1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3} \quad N_1 = 1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3} \tag{10}$$

$$N_3 = \frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3} \qquad N_4 = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2} \tag{11}$$

By replacing the approximate adopted solution in the governing equation, the following residual results

$$\varepsilon(x) = EI_\omega \frac{d^4\varphi_e}{dx^4} - GI_t \frac{d^2\varphi_e}{dx^2} + m_x(x) \neq 0 \tag{12}$$

that is used in Galerkin's functional, which must be minimized.

$$\begin{aligned} \Pi_i &= \int_0^\ell N_i(x) \cdot \varepsilon(x) dx = EI_\omega \int_0^\ell N_i(x) \frac{d^4\varphi_e}{dx^4} dx - \\ &- GI_t \int_0^\ell N_i(x) \frac{d^2\varphi_e}{dx^2} dx + \int_0^\ell N_i(x) m_x(x) dx = 0 \end{aligned} \tag{13}$$

The first two terms are integrated by parts

$$\begin{aligned} \Pi_i &= N_i(x)\left[GI_t \frac{d\varphi_e}{dx} - M_x(x)\right]\Bigg|_0^\ell + \frac{dN_i(x)}{dx} B_\omega(x)\Bigg|_0^\ell - \\ &- GI_t N_i(x) \frac{d\varphi_e}{dx}\Bigg|_0^\ell + EI_\omega \int_0^\ell \frac{d^2N_i(x)}{dx^2} \cdot \frac{d^2\varphi_e}{dx^2} dx + \\ &+ GI_t \int_0^\ell \frac{dN_i(x)}{dx} \cdot \frac{d\varphi_e}{dx} dx + \int_0^\ell N_i(x) \cdot m_x(x) dx \end{aligned} \tag{14}$$

When $\varphi_e(x)$ in substituted in the previous relation by its expression (8), the finite element force-displacement relation is obtained

$$\frac{EI_\omega}{\ell^3}\left(\begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} + \right.$$
$$\left. + k^2\ell^2 \begin{bmatrix} \frac{6}{5} & \frac{\ell}{10} & -\frac{6}{5} & \frac{\ell}{10} \\ \frac{\ell}{10} & \frac{2\ell^2}{15} & -\frac{\ell}{10} & -\frac{\ell^2}{30} \\ -\frac{6}{5} & -\frac{\ell}{10} & \frac{6}{5} & -\frac{\ell}{10} \\ \frac{\ell}{10} & -\frac{\ell^2}{30} & -\frac{\ell}{10} & \frac{2\ell^2}{15} \end{bmatrix}\right) \tag{15}$$

$$\cdot\begin{Bmatrix}\varphi_1\\ \varphi_1'\\ \varphi_2\\ \varphi_2'\end{Bmatrix}+m_x\begin{Bmatrix}\dfrac{\ell}{2}\\ \dfrac{\ell^2}{12}\\ \dfrac{\ell}{2}\\ -\dfrac{\ell^2}{12}\end{Bmatrix}=\begin{Bmatrix}M_{x_1}\\ B_{\omega_1}\\ M_{x_2}\\ B_{\omega_2}\end{Bmatrix}$$

where

$$k=\sqrt{\frac{GI_t}{EI_\omega}} \qquad (16)$$

In relation (16) E is the longitudinal modulus of elasticity for the material, G is the shear modulus of elasticity, I_ω and I_t are geometrical properties of the bar section, that is the warping moment of inertia and the St. Venant torsional moment of inertia, respectively.

The structural force-displacement relation is obtained by using the assembly procedure. In this process the boundary (support) conditions are imposed and finally, the nodal displacements, twisting moments and bi-moments at each mode can be evaluated.

3. FINITE ELEMENT DERIVED BY USING FIFTH DEGREE SHAPE FUNCTIONS

In comparison with the previous discussed finite element, this one, pictured in Fig. 2 is provided with three degrees of freedom at each node: the twisting angle φ and its two derivatives φ' and φ''.

In these circumstances the two vectors of nodal displacements and forces become:

$$d_e=\begin{Bmatrix}\varphi_1\\ \varphi_1'\\ \varphi_1''\\ \varphi_2\\ \varphi_2'\\ \varphi_2''\end{Bmatrix}\quad S_e=\begin{Bmatrix}M_{x_1}\\ B_{\omega_1}\\ 0\\ M_{x_2}\\ B_{\omega_2}\\ 0\end{Bmatrix} \qquad (17)$$

In this case the displacement field if approximated by a fifth degree polynomial:

$$\varphi_e(x)=\alpha_0+\alpha_1x+\alpha_2x^2+\alpha_3x^3+\alpha_4x^4+\alpha_5x^5 \qquad (18)$$

or shortly written:

$$\varphi_e=P^T\cdot\alpha$$

where

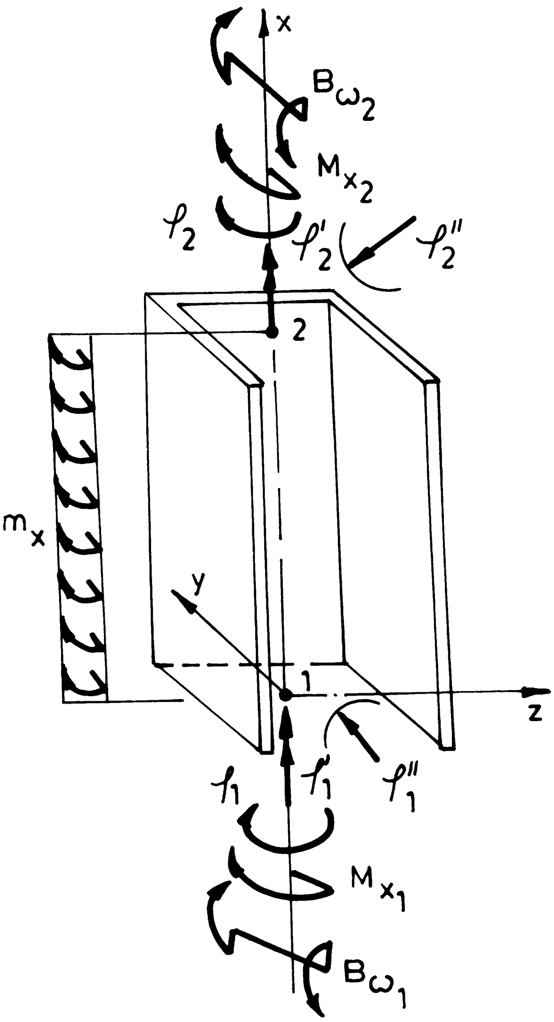

Figure 2.

$$P^T=\begin{bmatrix}1 & x & x^2 & x^3 & x^4 & x^5\end{bmatrix}$$
$$\alpha=\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5\end{bmatrix}^T \qquad (19)$$

By expressing the boundary conditions at the extremities of the finite element the vector of generalized coordinates $\{\alpha\}$ is obtained according to relation (3), but in the new case

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{10}{\ell^3} & -\frac{6}{\ell^2} & -\frac{3}{2\ell} & \frac{10}{\ell^3} & -\frac{4}{\ell^2} & \frac{1}{2\ell} \\ 15\ell^4 & \frac{8}{\ell^3} & \frac{3}{2\ell^2} & -\frac{15}{\ell^4} & \frac{7}{\ell^3} & -\frac{1}{\ell^2} \\ -\frac{6}{\ell^5} & -\frac{3}{\ell^4} & -\frac{1}{2\ell^3} & \frac{6}{\ell^5} & -\frac{3}{\ell^4} & \frac{1}{2\ell^3} \end{bmatrix} \tag{20}$$

The new form of the shape function matrix becomes

$$\begin{aligned} N_1 &= 1-\frac{10x^3}{\ell^3}+\frac{15x^4}{\ell^4}-\frac{6x^5}{\ell^5} \\ N_2 &= x-\frac{6x^3}{\ell^2}+\frac{8x^4}{\ell^3}-\frac{3x^5}{\ell^4} \\ N_3 &= \frac{x^2}{2}-\frac{3}{2}\frac{x^3}{\ell}+\frac{3}{2}\frac{x^4}{\ell^2}-\frac{1}{2}\frac{x^5}{\ell^3} \\ N_4 &= \frac{10x^3}{\ell^3}-\frac{15x^4}{\ell^4}+\frac{6x^5}{\ell^5} \\ N_5 &= -\frac{4x^3}{\ell^2}+\frac{7x^4}{\ell^3}-\frac{3x^5}{\ell^4} \\ N_6 &= \frac{1}{2}\frac{x^3}{\ell}-\frac{x^4}{\ell^2}+\frac{1}{2}\frac{x^5}{\ell^3} \end{aligned} \tag{21}$$

The same procedure is followed as in the previous case and after expressing the residual $\varepsilon(x)$ and Galerkin's functional, the finite element force-displacement relation can be written as:

$$\frac{EI_\omega}{\ell^3}\left(\begin{bmatrix} \frac{120}{7} & \frac{60\ell}{7} & \frac{3\ell^2}{7} & -\frac{120}{7} & \frac{60\ell}{7} & -\frac{3\ell^2}{7} \\ \frac{60\ell}{7} & \frac{192\ell^2}{35} & \frac{11\ell^3}{35} & -\frac{60\ell}{7} & \frac{108\ell^2}{35} & -\frac{4\ell^3}{35} \\ \frac{3\ell^2}{7} & \frac{11\ell^3}{35} & \frac{3\ell^4}{35} & -\frac{3\ell^2}{7} & \frac{4\ell^3}{35} & \frac{\ell^4}{70} \\ -\frac{120}{7} & -\frac{60\ell}{7} & -\frac{3\ell^2}{7} & \frac{120}{7} & -\frac{60\ell}{7} & \frac{3\ell^2}{7} \\ \frac{60\ell}{7} & \frac{108\ell^2}{35} & \frac{4\ell^3}{35} & -\frac{60\ell}{7} & \frac{192\ell^2}{35} & -\frac{11\ell^3}{35} \\ -\frac{3\ell^2}{7} & -\frac{4\ell^3}{35} & \frac{\ell^4}{70} & \frac{3\ell^2}{7} & -\frac{11\ell^3}{35} & \frac{3\ell^4}{35} \end{bmatrix}+\right.$$

$$\left.+k^2\ell^2\begin{bmatrix} \frac{10}{7} & \frac{3\ell}{14} & \frac{\ell^2}{84} & -\frac{10}{7} & \frac{3\ell}{14} & -\frac{\ell^2}{84} \\ \frac{3\ell}{14} & \frac{8\ell^2}{35} & \frac{\ell^3}{60} & -\frac{3\ell}{14} & -\frac{\ell^2}{70} & \frac{\ell^3}{210} \\ \frac{\ell^2}{84} & \frac{\ell^3}{60} & \frac{\ell^3}{630} & -\frac{\ell^2}{84} & -\frac{\ell^3}{210} & \frac{\ell^4}{1260} \\ -\frac{10}{7} & -\frac{3\ell}{14} & -\frac{\ell^2}{64} & \frac{10}{7} & -\frac{3\ell}{14} & \frac{\ell^2}{84} \\ \frac{3\ell}{14} & -\frac{\ell^2}{70} & -\frac{\ell^3}{210} & -\frac{3\ell}{14} & \frac{8\ell^2}{35} & -\frac{\ell^3}{60} \\ -\frac{\ell^2}{84} & \frac{\ell^3}{210} & \frac{\ell^4}{1260} & \frac{\ell^2}{84} & -\frac{\ell^3}{60} & \frac{\ell^4}{630} \end{bmatrix}\right)+ \tag{22}$$

$$+m_x\begin{Bmatrix} \frac{\ell}{2} \\ \frac{\ell^2}{10} \\ -\frac{\ell^3}{120} \\ \frac{\ell}{2} \\ -\frac{\ell^2}{10} \\ \frac{\ell^3}{120} \end{Bmatrix}=\begin{Bmatrix} M_{x_1} \\ B_{\omega_1} \\ 0 \\ M_{x_2} \\ B_{\omega_2} \\ 0 \end{Bmatrix}$$

4. CONCLUSIONS

The analysis of thin-walled bars subjected to torsion is a frequently met problem in the structural design.

In order to eliminate the restrictions imposed in order to find the analytical solutions of the governing equation, the finite element method is approached.

In this method there are several possibilities of adopting adequate finite elements and among them, two types of linear finite element are considered.

Generally, the higher order of shape functions yields to more accurate results of the analysis.

Bibliography

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