# STOCHASTIC OPTIMAL CONTROL OF A TWO-DIMENSIONAL DYNAMICAL SYSTEM

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We consider the following controlled two-dimensional dynamical system:

$$\dot{x}(t) = -k\,x(t)y(t),$$
$$\dot{y}(t) = k\,x(t)y(t) + f[x(t), y(t)] + b[x(t), y(t)]u(t) + \{v[x(t), y(t)]\}^{1/2}\,\dot{B}(t),$$

where $k$ is a positive constant, $u(t)$ is the control variable, $v[x(t),y(t)]$ is a positive function and $B(t)$ is a standard Brownian motion.

Let $x(0) = x$ and $y(0) = y$ be such that $0 < x + y < d$, and define the *first-passage time*

$$T(x,y) = \inf\{t \geq 0 : x(t) + y(t) = 0 \text{ or } d\}.$$

Our aim is to find the value $u^*$ of the control variable that minimizes the expected value of the cost criterion

$$J(x,y) = \int_0^T \left\{\tfrac{1}{2} q[x(t), y(t)]u^2(t) + \lambda\right\} dt,$$

where $q[x(t), y(t)]$ is a positive function and $\lambda$ is a *negative* constant. Hence, the aim is to maximize the expected survival time in the interval (0, $d$), taking the quadratic control costs into account.

This type of optimal control problem, for which the final time is a random variable, has been termed LQG homing by Whittle (1982). These problems are generally very difficult to solve explicitly, especially in two or more dimensions. LQG homing problems have been considered, in particular, by Lefebvre and Zitouni (2014) and Makasu (2013), who solved explicitly a two-dimensional problem.

In this paper, exact and explicit solutions will be found in particular cases by making use of the *method of similarity solutions* to solve the partial differential equation satisfied by the value function.

**Keywords:** *Dynamic programming, Brownian motion, first-passage time, partial differential equations, error function.*

## References

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