TOPOLOGICAL MATERIALS AND TOPOLOGICAL QUANTUM COMPUTING

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Abstract — **A paradigm to build a quantum computer based on topological invariants is highlighted. The identities in the ensemble of knots, links and braids originally discovered in relation to topological quantum field theory are generalized to define Artin braid group—the mathematical basis of topological quantum computation (TQC). Vector spaces of TQC correspond to associated strings of particle interactions and TQC operates its calculations on braided strings of special physical qvasiparticles—anyons with non-Abelian statistics. The physical platform of TQC is to use the topological quantum numbers of small groups of anyons as qubits and to perform operations on these qubits by exchanging the anyons, both within the groups that form the qubits and, for multi-qubit gates, between groups. By braiding two or more anyons, they acquire up a topological phase or Berry phase similar to that found in the Aharonov-Bohm effect. Topological matter such as fractional quantum Hall systems and novel discovered topological insulators open the way to form system of anyons—Majorana fermions— with the unique property of encoding and processing quantum information in a naturally fault-tolerant way. In the topological insulators due to its fundamental attribute of topological surface state occurrence the bound Majorana fermions are generated at its heterocontact with superconductors.One of the key operations of TQC—braiding of non-Abelian anyons— it is illustrated how can be implemented in one-dimensional topological isolator wire networks.**

Index Terms — **topological quantum computation, braided strings, vector spaces, qubits, anyons, topological phase, topological insulators, Majorana fermions, braiding, wire networks.**

I. INTRODUCTION

Quantum mechanics was elaborated since the late 1920s, and it can be safely said that without it our modern technological society would not exist. Quantum effects power almost all that we know, from transistors and lasers to medical imaging nuclear weapons and more. However, only in the last 10-15 years has it been realized that the most fundamental attributes of quantum mechanics - ones not presently used in any technology - can be configured for quantum computing and communication. Formally, a quantum computation is performed through a set of transformations, called gates [1]. Like quantum operators a gate applies unitary transformation U to a set of qubits in a quantum state $|\psi>$. At the end of the calculation, a measurement is performed on the qubits (which are in the state $|\psi'> = U |\psi>$). There are many ways to choose sets of universal quantum gates. These are sets of gates from which any computation can be constructed, or at least approximated as precisely as desired. Such a set allows one to perform any arbitrary calculation without inventing a new gate each time. The implementation of a set of universal gates is therefore of crucial importance. It can be shown that it is possible to construct such a set with gates that act only on one or two qubits at a time. Ordinary quantum computation uses simple quantum two level systems (e.g. electron or nuclear spins, atomic hyperfine states, etc.) as quantum bits ('qubits') with one- and two-qubit unitary operations serving as universal quantum gates.

The great promise of quantum computers has to be balanced against the great difficulty of actually building them. Foremost among the difficulties is the fundamental challenge of defeating decoherence and errors. Small improvements to current strategies are not sufficient to overcome this problem; radically new ideas are required. Ordinary quantum computation uses simple quantum two level systems (e.g. electron or nuclear spins, atomic hyperfine states, etc.) as quantum bits ('qubits') with one- and two-qubit unitary operations serving as universal quantum gates. The main problem is quantum decoherence, the inevitable continuous dephasing of a quantum state due to its interaction with the environment. A new paradigm is to build a quantum computer which is topologically immune to quantum decoherence and such platform is called topological quantum computation (TQC)[1]. The idea is to use the topological quantum numbers of small groups of anyons as qubits and to perform operations on these qubits by exchanging the anyons, both within the groups that form the qubits and, for multi-qubit gates, between groups. Anyons are unusual special type quasiparticles unlike the electrons and protons and having the desired mathematical properties. The importance of such a paradigm is that it allows one to make direct contact with the circuit model of quantum computation and it enables algorithmic questions to be tackled independently of the details of experimental implementation, at least initially. TQC employs many-body physical systems with the unique property of encoding and processing quantum information in a naturally fault-tolerant way. Research on topological quantum computation has become a highly interdisciplinary field, with frontiers in physics, mathematics, and computer science. Moreover, advances in the theoretical understanding of abstract topology, in physical realizations of topological matter, and in computational paradigms have been closely interrelated [2], with developments in one area strongly influencing the others. Recent years have witnessed significant theoretical and experimental developments. These include major experimental and theoretical advances in fractional quantum Hall systems that support the existence of non-Abelian anyons—the building block of topological quantum computation—as well as the predication and experimental discovery of novel spin–orbit systems such as topological insulators.

The aim of the paper is to review together the latest developments in a TQC and topological materials with the goal of underlining the synergy between computer and material sciences approaches.

II. MATHEMATICAL CONSIDERATIONS ON TOPOLOGICAL COMPUTATION

Knots, links and braids are the mathematical basis of TQC. A representation of a knot was defined to be a closed polygonal curve in space. Links are then a combination of knots that are intertwined. It was not until later (1920s) that mathematicians became interested in representations of braids which were defined to be a set of n polygonal curves stretching from z= 0 plane (in R3) to the z = 1 plane where the kth curve stretches from (1/2, k/n, 0) to (1/2,k/n, 1) and the z value is strictly increasing and the curves do not intersect [3]. Braids clearly have some algebraic properties. There is a clear identity braid, which is just formed by connecting the start and end points with straight lines. We can imagine"adding" two braids with the same number of strands. This addition will be associative" a(bc) = (ab)c. Similarly, we could imagine by exactly reversing the way we did the braiding, that we could add two braids which could be manipulated to obtain the identity (an inverse braid). Finally, if we add many braids together it is clear it will still remain a braid. Thus a group - Artin braid group - is obtained for which we can establish about how it may be generated and what equalities are required for combinations of those generators so that we can determine if two braids are equivalent [3]. Let to define k as the exchange of the k^{th} curve with $(k+1)^{st}$ curve where the k^{th} curve passes over the $(k+1)^{st}$ (Fig 1). It is easy to observe the following set of identities:

$\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| < 3$; $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

The first of equations indicates that two disjoint exchanges are commutative and the second can be seen in Fig.1.

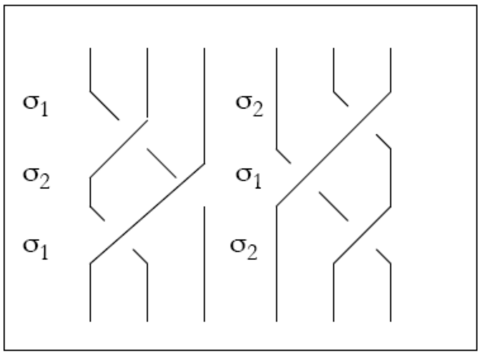

Fig.1. *Two braids continuously deformed into each other without cutting any of the strands illustrating the second equality.*

The abovementioned conditions are all which are required to define general braid group.

In TQC vector spaces corresponding to associated strings of particle interactions are interrelated by recoupling transformations that generalize the usual QC mapping. A full representation of the Artin braid group on each space is defined in terms of the local interchange phase gates and the recoupling transformations [1]. These gates and transformations have to satisfy a number of identities in order to produce a well-defined representation of the braid group. These identities were discovered originally in relation to topological quantum field theory.

At first sight, a TQC does not seem much like a computer at all. It works its calculations on braided strings—but not physical strings in the conventional sense. Rather, they are what physicists refer to as world lines, representations of particles as they move through time and space. However TQC are based on non-Abelian statistics and a special type of particles –anions - are required, which can appear in physical systems as the result of many-body interactions.

III. TOPOLOGICAL COMPUTATION WITH ANIONS

In conventional computing zeroes and ones are created by switching an electric current on and off in a MOS transistor. Ordinary quantum computation uses simple quantum two level systems (e.g. electron or nuclear spins, atomic hyperfine states, etc.) as quantum bits ('qubits') with one- and two-qubit unitary operations serving as universal quantum gates. Physical basis of TQC is more subtle and use the anions - excitations in a two-dimensional electronic system that behave a lot like the particles and antiparticles of high-energy physics. They are able to carry charges that are fractions of the fundamental charge of the electron. The spin of these quasiparticles can take on any real value. This is of course related to their statistics and the fact that they are neither fermions nor bosons. There are no physical processes that can create or destroy isolated anyons. This is important if we intend to use them in a quantum computer. If the anyons could spontaneously appear or disappear any quantum operation using them would fail. They also have antiparticles, which they can interact with to combine or annihilate. Anyons can also combine with other anyons that are not their antiparticle. Fundamental attribute of the a group of anyons is its quantum toplogical number.

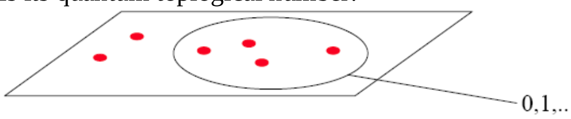

Fig.2. *Groups of particles have quantum numbers*

By braiding two anyons, they acquire up a topological phase similar to that found in the Aharonov-Bohm effect – that is the phase given to a charged particle accumulates when it travels around a solenoid. Just like the phase obtained in Aharonov-Bohm effect, the phase only depends on how many times the anyons wrap around each other and not the path they follow. In the one dimensional representation of the braid group, we obtain $\sigma j = e^{i \theta j}$ for identical anyons, where θj is the topological phase added by the σ_j operation [2].

Alternatively, we could have a multidimensional representation, which allow us to have nonabelian anyons as well. These nonabelian anyons are more useful for quantum computing than abelian anyons. We now must consider how anyons can combine and split. Each model of anyons will have different fusion rules. The fusion rules determine the total charge, c, when a and b combine. These are written as a x b = ΣcNcab. where Nc ab is a nonnegative integer and the sum is over the complete set of labels of the composite. The composition rules are symmetric (a x b = b x a) so the possible charges do not depend on which side the anyon came from. Note that if

Ncab is zero the charge, c, cannot be formed, while if it is one there is a unique way of obtaining c, and Ncab can also be greater than one. So Ncab represents the number of distinguishable ways that a charge c can be obtained. The distinguishable ways that a and b can be combined to form c then represents an orthonormal basis for a Hilbert space, which is called a fusion space.

The next idea that is introduced is the R matrix, which is the braid operator, and the F matrix, which is the fusion operator. These are each specific to a given model. The last result of the formalism of anyons that should be noted is that the Hilbert space can be shown to be exponential in size making it a good space to do quantum computations.

Following step is to use the topological quantum numbers of small groups of anyons as qubits and to perform operations on these qubits exchanging the anyons, both within the groups that form the qubits and, for multi-qubit gates, between groups. Summary of TQC basis includes:

• *Uses 2D systems which have quasiparticles with NonAbelian Statistics.*
• *Quantum Information is encoded in nonlocal topological degrees of freedom that do not couple to any local quantity.*
• *States can be manipulated by dragging (braiding) quasiparticles around each other.*
• *The operations (gates) performed on the qubits depends only on the topology of the braids.*

IV. TOPOLOGICAL STATE OF CONDENSED MATER

Depending on the electronic band structure and transport characteristics uncountable number of materials and substances can be classified quite simply in terms of their conductive behavior into one of three types — insulators, semiconductors and metals. More than three decade ago there was established that spin-orbit interaction (SOI) has an important pattern on band structure of solid state matter. Among different qualitative features induced by SOI the band inversion of electronic spectrum near the Fermi level has been discovered. Such type of electronic spectrum was identified in different type of semimetalic and narrow-gap semiconductors $Bi_{1-x}Sb_x$, $Pb_{1-x}Sn_xTe$, Bi_2Te_3, HgTe, $TlBiTe_2$ etc. In the context of low dimensional structure investigations the band spectrum inversion was shown to generate new type of interface gapless states with linear spectrum at the heterocontact boundaries. Last years investigations [4] have reopened the interest to materials with inverted band spectra. Due to new type of the symmetry break like that characteristic for the integer and fractional quantum Hall effects the electronic states was shown to have topological nature and materials have been named toplogical insulators (TI). Thus in TI a new state of matter appear, distinguished from a regular band insulator by a nontrivial time-reversal topological invariant, which characterizes its band structure, and non-trivial interplay of charge and spin degree of freedom of band electrons. In results new physics and phenomena related to this states have greatly emerged. Several of such new TI properties are reviewed in the paper as well as some old observed properties of materials with band inversion. Many intriguing properties of TI can be ascribed to the existence of two-band gaplees Dirac electrons in its low-energy band structure. Actually, Dirac electrons with finit gap in materials have a long history starting from bismuth that has three-dimensional massive Dirac electrons in its band structure

The most robust observable consequence of a nontrivial topological character of these materials is the presence of gapless helical edge states (interface states of inverted heterocontacts), whose gapless states is protected by time-reversal symmetry and is thus robust to perturbations that do not break this symmetry (Fig.3). Like the Hall state the “bulk” of the electron gas of TI is an insulator, but along its suface, the states can be gapless. Within a certain parameter range the surface states of TI are well described by a Dirac cone, allowing for parallels with graphene and relativistic physics, and prohibiting backscattering. A prerequisite for such experiments is a highly tunable surface state which is decoupled from the residual bulk carriers. Despite considerable recent evidence of TI surface states in ARPES and STM, transport experiments are complicated due to significant parallel conduction through bulk states, limited surface density tunability, and uncertainty

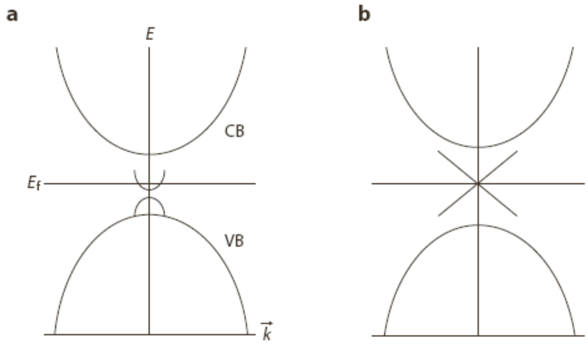

Fig.3. *The electronic band structures of topological insulators, a new class of quantum matter with (a) a robust metallic state at the surface/edge and insulating properties in the bulk/surface, and (b) a conductive state at the surface or edge with zero gap and the same linear energy dispersion as graphene.*

of the surface to bulk coupling. The unusual planar metal that forms at the surface of topological insulators inherits topological properties from the bulk bandstructure. The manifestation of this bulk-surface connection occurs at a smooth surface where momentum along the surface remains well-defined: in its simplest possible form, each momentum along the surface has only a single spin state at the Fermi level, and the spin direction rotates as the momentum moves around the Fermi surface ensuring a non-trivial Berry’s phase. These two defining properties of topological insulators namely spin-momentum locking of surface states and π Berry’s phase along with the consequences such as the robustness to non-magnetic disorder could be most clearly demonstrated with the discovery of the second generation of topological insulators. At the same time the spectrum and characteristics of topological surface states (TSS) depending on geometrical configuration can be manipulates by different factors: electrical and magnetic fields, strain and deformation ets. For this reason TI are being explored with a view towards applications, as a potential platform for TQC [4].

V. MAJORANA FERMIONS OF TI AS ANIONS FOR TOPOLOGICAL COMPUTATION

In condensed matter physics, Majorana fermions can arise due to a paired condensate that allows a pair of fermionic quasiparticles to “disappear" into the condensate. They have been predicted in a number of physical systems. At the edge of a superconductor [2] there may or may not be discrete states within the energy gap that are bound to the boundary. Such a quasiparticle is its own antiparticle and they are defined as Majorana fermion. It is essentially half of an ordinary Dirac fermion. Due to the particle-hole redundancy, a single fermionic state is associated with each pair of E energy levels. The presence or absence of a fermion in this state dfines a two level system with energy splitting E. Majorana zero modes must always come in pairs (for instance, a 1D superconductor has two ends), and a well separated pair defines a degenerate two level system, whose quantum state is stored nonlocally. Majorana bound states defines a degenerate two level system - a qubit. Importantly, the quantum information in the qubit is stored non locally. The state can not be measured with a local measurement on one of the bound states. This is crucial, because the main dfficulty with making a quantum computer is preventing the system from accidentally measuring itself. 2N Majorana bound states defines N qubits - a quantum memory.

In two dimensions a number of chiral Majorana edge modes can appear, which resemble chiral modes in the quantum Hall effect, but for the particle-hole redundancy. A spinless superconductor with px +ipy symmetry is the simplest model 2D topological superconductor. Such superconductors will also exhibit Majorana bound states at the core of vortices.

Combining topological insulators with ordinary superconductors leads to an exquisitely correlated interface state that, like a topological superconductor, is predicted to host Majorana fermion excitations and its properties has proposed to be for fault tolerant quantum information processing [4].

Majorana fermion can be created in several ways using topological insulators [4]. The most direct proposal using a 3D topological insulator is to consider the proximity effect from an ordinary s-wave superconductor. A magnetic vortex core in such a system will carry a zero-energy Majorana fermion state localized near the vortex in the interface layer, as well as possibly ordinary electronic modes (Fig.5).

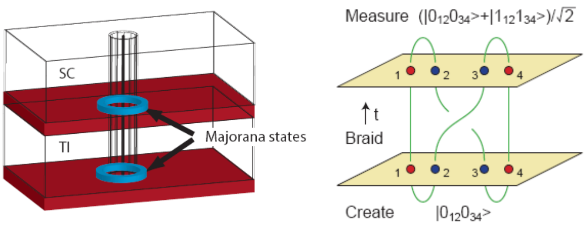

Fig. 4. *Majorana bound stats in topological isolator/superconductor heterojunction and its braiding.*

There are analogous ways to create a Majorana fermion using strong spin-orbit quantum wells rather than topological insulators.

Recently, a network of 1D semiconductor quantum wires has been proposed [5] as a suitable platform to create, transport, and fuse Majorana fermions at the wire ends. The wire network consists of wire segments in the TS state (shown in red with numbers in Fig. 5) connected by segments in the non-topological superconducting (NTS) state (shown in blue without numbers in Fig. 5).

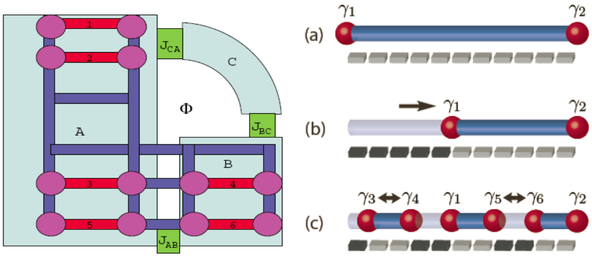

Fig. 5. *Schematic of entanglement generation and manipulation in quantum wire topological qubits using superconductor Josephson junctions.*

Local gates allow Majorana fermions to be transported, created, and fused as outlined in Fig. 5. As one germinates pairs of Majorana fermions, the ground state degeneracy increases as does our capacity to topologically store quantum information in the wire. Specifically, 2n Majoranas generate n ordinary zero-energy fermions whose occupation numbers specify topological qubit states. Adiabatically braiding the Majorana fermions would enable manipulation of the qubits, but is not possible in a single wire [5]. The Majorana fermion states are transported by shifting the end points of the TS segments by applying locally tunable external gate potentials (which control μ).

VI. CONCLUSIONS

There is a great deal of progress that has been made in the theory of topological quantum computing. Anyons and its braids in the topological matter such as fractional quantum Hall systems and novel discovered topological insulators excellent simulate quantum gates to arbitrary accuracy. Combining topological insulators with ordinary superconductors leads to an exquisitely correlated interface state that, like a topological superconductor, is predicted to host Majorana fermion excitations and its properties has proposed to be for fault tolerant quantum information processing.

A network of topological insulator quantum wires in the vicinity of an *s*-wave superconductor allows universal TQC. Such approach enable the Majorana fusion rules to be probed, along with networks that allow for efficient exchange of arbitrary numbers of Majorana fermions.

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