Newton's Method for Solving Quadratic Programming Problems with Simple Constraints

Vasile MORARU[1], Dumitru STOIAN[2]
[1] Technical University of Moldova, [2]State University Alecu Russo from Bălţi
moraru@mail.utm.md, dmitrii.stoian@gmail.com

***Abstract* — In this paper the convex quadratic programming problem with simple constraints, following judicious transformations of Karush-Kuhn-Tucker optimality conditions, is reduced to solving a system of smooth equations. In order to solve the obtained system of equations, the Newton's method, which provides a quadratic speed of convergence, is applied.**

***Keywords* - Karush-Kuhn-Tucker conditions, Newton's method, nonlinear complementarity functions.**

I. INTRODACTION

The paper considers the problem of quadratic programming with simple constraints:

$$\left.\begin{array}{c} f(x)=\frac{1}{2}x^T Hx+g^T x\rightarrow \min \\ \text{subject to}: x\geq 0, \end{array}\right\} \quad (1)$$

where $H\in\Re^{n\times n}$ is a symmetric and positive definite matrix: $x^T Hx>0, \forall x\in\Re^n$, and, $g\in\Re^n$. The symbol "T" indicates the operation of transpose.

The problem considered has various real practical applications: structural analysis, VLSI design, support of vector machines, graph theory, etc. A complete bibliography on general issues of quadratic programming can be found in [1], a work containing over 1000 (one thousand) of references!

One of the most popular techniques for solving optimization problems are primal-dual methods based on solving Karush-Kuhn-Tucker conditions of optimality [2-5]. A vector $x^*\in\Re^n$ is an optimal solution for problem (1) if and only if the Lagrange multiplier, which verifies the Karush-Kuhn-Tucker algebraic relations, exists [2]:

$$\left.\begin{array}{l} \nabla f(x^*)-\lambda^*=0, \\ x^{*T}\nabla f(x^*)=0, \\ x^*\geq 0, \lambda^*\geq 0, \end{array}\right\} \quad (2)$$

The system of equations and inequalities (2) with help of the complementary functions [6] can be reduced to a system composed only of equations, so that the Newton's method of solving can be applied. A function $\varphi:\Re\times\Re\rightarrow\Re$ is called complementary function if the set of solutions for the equation $\varphi(a,b)=0$ coincides with the set of solutions of the system: $ab=0$, $a\geq 0, b\geq 0$.

In the paper [7], the function of Evtushenco was considered as φ [8]: $\varphi(a,b)=\frac{1}{4}\left\{(a-b)^2-(a+b)|a+b|\right\}$.

This function unlike other complementary functions [6] is a smooth function for $\forall a,b\in\Re$. A similar technique of transformation of a system (2) was presented in the paper [9], where function $(a)^+=\max\{0,a\}$ has been used, which is an undifferentiated function.

Subsequently, we will apply the proposed technique in the paper [10], showing that we can reduce the system (2) to another convenient consisting only of n equations with n unknown. This gives us a great advantage in case of solving mathematical programming problems in large dimensions.

II. REFORMULATION OF KARUSH-KUHN-TUCKER CONDITIONS

Consider two functions $u(x)$ and $v(x):\Re\rightarrow\Re_+$ defined as follows [10]:

$$u(x)=x^2\max(0,x)=\frac{1}{2}\left(x^3+|x|x^2\right),$$

$$v(x)=-x^2\min(0,x)=-\frac{1}{2}\left(x^3-|x|x^2\right).$$

The above defined functions satisfy the following properties [10]:

P1. $u(x)=\begin{cases}=0, & \forall x\leq 0,\\ >0, & \forall x>0,\end{cases}$ $\quad v(x)=\begin{cases}>0, & \forall x<0,\\ =0, & \forall x\geq 0.\end{cases}$

P2. Functions $u(x)$ and $v(x)$ are continuously twice differentiable for $\forall x\in\Re$.

P3. $u(x)=0$ şi $v(x)=0$ if and only if $u'(x)=0$, respectively $v'(x)=0$ for any $x\neq 0$.

P4. $u(x)\times v'(x)=u'(x)\times v(x)=0$ for any $x\neq 0$.

Let $y=(y_1,y_2,\ldots,y_n)^T$ be the vector of the auxiliary variables $y_i, i=1,2,\ldots n$. We define the operators:

$$U(y)=\left(u(y_1),u(y_2),\ldots,u(y_n)\right)^T:\Re^n\rightarrow\Re^n,$$

$$V(y)=\left(v(y_1),v(y_2),\ldots,v(y_n)\right)^T:\Re^n\rightarrow\Re^n.$$

Using functions $u(x)$ and $v(x)$, based on the properties **P1-P4**, the Karush-Kuhn-Tucker system (2) can be transformed in the following equivalent system of $2n$ nonlinear smooth equations with $2n$ unknown:

$$\left.\begin{array}{l} \nabla f(x)-U(x)=0, \\ x-V(y)=0. \end{array}\right\}$$

From last equation we have $x=V(y)$ and so, taking in consideration that $\nabla f(x) = Hx + g$, we get a system of n equations with many unknowns $y_1, y_2, \ldots, y_n$:

$$HV(y) - U(y) = -g . \qquad (3)$$

We note as $U'(y)$ and $V'(y)$ the diagonal matrices of $n \times n$ dimension with $u'(y)$ elements, respectively $v'(y)$, $i = 1,2,\ldots,n$:

$$U'(y) = diag(u'(y_1), u'(y_2), \ldots, u'(y_n)),$$
$$V'(y) = diag(v'(y_1), v'(y_2), \ldots, v'(y_n)).$$

With these notations the Jacobian matrix of $F(y) = HV(y) - U(y) + g$ becomes
$F'(y) = HV'(y) - U'(y)$.

Theorem. The Jacobian matrix $F'(y)$ is not singular in the neighbourhood of the system solution (3).

Proof. Let any $p \in \Re^n$. Than, from $F'(y)p = 0$ we have:

$$HV'(y)p = U'(y)p \qquad (4)$$

Whence

$$[HV'(y)p]^T V'(y)p = [U'(y)p]^T V'(y)p =$$
$$= [V'(y)U'(y)p]^T p = 0$$

because $V'(y)U'(y) = O \in \Re^{n \times n}$.

On the other hand,

$$0 = [HV'(y)p]^T V'(y)p \geq \mu \|V'(y)p\|_2^2 > 0,$$

where $\mu > 0$ is the lowest eigenvalue of the matrix *H*.

Thus $V'(y)p = 0$. From (4) it follows that also $U'(y)p = 0$. Thus, if $v'(y_s) \neq 0$ results $p_s = 0$ and $u'(y_s) = 0$. If $v'(y_s) = 0$ it results that $u'(y_s) \neq 0$ and $p_s = 0$. Hence $p_s = 0, \forall s = 1,2,\ldots n$, namely $rang(F'(y)) = n$.

The theorem is proved.

III. THE NEWTON'S METHOD

The above theorem guarantees that the system of equations (3) can be solved using Newton's method, which has superlinear speed of convergence in the neighbourhood of the solution.

Let $y^{(k)}$ be the current approximation of the solution y^*. Than the approximation $y^{(k+1)}$ is obtained by solving the system of linear equations: $F'(y^{(k)})(y - y^{(k)}) = -F(y^{(k)})$, or, in expanded form, of the system:

$$[HV'(y^{(k)}) - U'(y^{(k)})](y - y^{(k)}) =$$
$$= -HV(y^{(k)}) + U(y^{(k)}).$$

The solution considered is the limit of the string

$$\{x^{(k+1)}\} = \{V(y^{(k+1)})\}.$$

Numerical example.

$$f(x) = (x_1 + x_2)^2 + \frac{1}{9}(x_2 - x_1 - 10)^2 + (x_3 + 5)^2 .$$

Here

$$H = \begin{pmatrix} \frac{20}{9} & \frac{16}{9} & 0 \\ \frac{16}{9} & \frac{20}{9} & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} -\frac{20}{9} \\ -\frac{20}{9} \\ -10 \end{pmatrix}.$$

The matrix H has eigenvalues $\mu_1 = 4/9$, $\mu_2 = 2$, $\mu_3 = 4$, and so is positive defined. System (3) has the form

$$\left.\begin{array}{l} -\frac{29}{18} y_1^3 - \frac{8}{9} y_2^3 + \frac{11}{18} y_1^2 |y_1| + \frac{8}{9} y_2^2 |y_2| = -\frac{20}{9}, \\ -\frac{8}{9} y_1^3 - \frac{29}{18} y_2^3 + \frac{8}{9} y_1^2 |y_1| + \frac{11}{18} y_2^2 |y_2| = \frac{20}{9}, \\ -\frac{3}{2} y_3^3 + \frac{1}{2} y_3^2 |y_3| = -10. \end{array}\right\}$$

Solution: $y_1^* = 1.5874; y_2^* = -1.0000; y_3^* = 2.1544;$
$x_1^* = 0.0; x_2^* = 1.0; x_3^* = 0.0$.

IV. CONCLUSIONS

The procedure proposed for the transformation of the Karush-Kuhn-Tucker system allows solving of the quadratic programming problem with simple restrictions in large dimensions. The main element is represented by the use of the functions $u(x)$ and $v(x)$, which helps to reduce the problem to a system in convenient dimensions that can be effectively solved by means of Newton's method.

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